

MODELING PRIVATE EQUITY FUNDS AND PRIVATE EQUITY COLLATERALISED FUND OBLIGATIONS

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The recent development of the securitisation of funds of private equity funds poses the question of the individual and joint modelling of the underlying funds. Private equity funds are different from other managed funds because of their particular bounded life cycle: when the fund starts, the investment partners make an initial capital commitment, the fund managers gradually draw down the committed capital into investments, returns and proceeds are distributed as the investments are realised and the fund is eventually liquidated as the final investment horizon is reached. Modelling private equity funds therefore requires three stages: the modelling of the commitment drawdowns, the modelling of the investment value and the modelling of the return repayments. A standard lognormal process is utilised for the dynamics of the investment value. Squared Bessel processes are utilised for the dynamics of the rates of drawdowns and repayments.

Keywords: Private equity fund; fund of funds; collateralised fund obligation (CFO); Cox–Ingersoll–Ross (CIR) process; squared Bessel process.

Résumé

Le récent développement de la titrisation de fonds de fonds de placements privés pose la question de la modélisation individuelle et jointe des fonds sous-jacents. Les fonds de placements privés sont différents des autres sociétés d'investissement à cause de leur cycle de vie particulier et limité: au démarrage du fonds, les associés s'engagent sur un apport initial en capital; puis les gérants du fonds opèrent des tirages progressifs sur le capital apporté pour procéder à des investissements; les revenus et les profits sont distribués à mesure que les investissements sont réalisés; enfin, le fonds est liquidé lorsque l'horizon d'investissement est atteint. La modélisation d'un fonds doit donc se faire en trois étapes: la modélisation des tirages sur l'apport en capital, la modélisation de la valeur des investissements et enfin la modélisation des paiements et remboursements des dividendes et retours sur investissements. Un processus lognormal standard est utilisé

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pour la dynamique de la valeur des investissements. Des processus de Bessel carré sont utilisés pour la dynamique des taux de tirage et de remboursement.

Mots clés: Fonds de placements privés; fonds de fonds; fonds commun de fonds (FCF); processus de Cox-Ingersoll-Ross (CIR); processus de Bessel carré.

1. Introduction

A private equity fund (PEF) is a pooled investment vehicle whose purpose is to negotiate privately purchases of common and preferred stocks, subordinated debt, convertible securities, warrants, futures and other securities of companies that are mostly unlisted. For a very good introduction on the subject of PEFs, refer to Moise *et al.* [1]. To the author's knowledge, there is yet no mathematical and financial literature about this subject. The main technical characteristics of this investment vehicle are detailed hereafter.

PEFs are organised as follows. The general partners (GPs) of the fund are responsible for making investment decisions on behalf of the investors, who are limited partners (LPs). Terms and conditions of the fund are agreed by contract, which, among other elements, specifies the capital committed by the LPs, the commitment period and the legal maturity of the fund. The commitment period is the period of time, usually several years, over which the fund may draw down committed capital from the LPs. Capital drawdowns are unscheduled and can occur at any time within the commitment period but cannot exceed the total committed capital initially made by any given LP to the fund. The legal maturity of a fund is usually in the order of ten years, by which time the fund is liquidated for cash distributions.

Funds can be categorised by sector (see Table 2), by strategy and asset class (see Table 3) and/or by industry. This allows the assignment of a certain risk/return profile to the asset portfolio. However, as said before, investments are generally made on unlisted companies, which makes the exercise of valuing the fund tricky. Besides, reporting the net asset value (NAV) of a PEF is the responsibility of the GPs who can keep the portfolio at its booking value for a while before revealing mark-ups or mark-downs. As a result, because the reported NAV is not always meaningful or reliable, what counts the most is the "exit" cash flows that are effectively realised and will take place for example through sales, mergers, initial public offerings or at the completion of the fund investment horizon when all assets have been liquidated and the proceeds repaid to the LPs. Thus, the performance of a fund tends to be evaluated on a cash flow basis using the internal rate of return (IRR) measure.

Further difficulties arise when valuing a PEF. Realised gains are subject to tax before being repaid. Compensation fee to the GPs, generally based on the portfolio returns, must also be paid. Besides, the investment process entails substantial uncertainty in terms of timing of both drawdowns and repayments, which are essentially unpredictable, because they are mainly driven by market opportunities.

Indeed, private equity partnerships are peculiar in that they are never actually fully invested. The funds collect capital from investors when attractive investments arise while simultaneously paying out realised returns on previously invested capital when the GP feels it is time to sell or has an opportunity to do so. Typically, in a PEF, the cash investments occur over four to six years whilst the cash returns start occurring after a couple of years. To overcome the initial inefficient capital use and to ensure optimum investment in private equity, the LP may adopt an “over-commitment” strategy, in which he commits more capital to the fund than he has at his disposal, in the expectation that earlier distributions received can be used to fund later drawdowns. In practice, a credit facility is often in place to provide liquidity equal to the over-commitment in the event that liquid reserves are insufficient to meet a capital drawdown.

Investors can also gain access to a variety of PEFs by investing in a private equity fund of funds. In a fund of funds, capital from a number of investors is aggregated for investment in many underlying funds (generally between ten and fifty). This gives the investor the opportunity to diversify his risk not only across geographic locations and industries but also across a variety of fund managers, fund sectors (see Table 2), investment horizons and investment cycles. Indeed, investment in funds of different “vintages” (inception years of the funds) and maturities limits the exposure of a fund of funds to adverse economic cycles.

Securitisation of funds of private equity funds is the last and most recent stage in the asset securitisation business. Equity fund securitisation is alternatively referred to as collateralised equity obligation (CEO) or collateralised fund obligation (CFO). Given that CFOs can be built from the collateralisation of either private equity funds or public hedge funds and that this note refers to PEFs only, we will refer to this instrument as “private equity CFO”. For more about the mechanism behind private equity CFOs, see the reports by Moise *et al.* [1] or Cheung *et al.* [2]. The principle behind a private equity CFO is actually the same as for standard asset-backed securities. A special purpose vehicle (SPV) is set up that issues rated and non-rated fixed-income bonds or floating-rate notes to investors. The proceeds are utilised to purchase a portfolio of PEFs that serves as a collateral for the interest and principal payments on the notes.

From the SPV standpoint, the key risk associated with a private equity CFO lies in the timing mismatches between its cash flow liabilities and the cash flows produced by its assets. To circumvent the very high variability of cash flow distributed by PEFs, it is common for the SPV sponsor to put in place a reserve account that is set aside at the closing of the note offering and will cover some of the initial expenses and coupon payments or to contract with a third party to provide a liquidity facility. Of course, all this is at a cost for the SPV and eventually for the note investors but is helpful in mitigating the liquidity risk.

From the note holder’s standpoint, the liquidity risk associated with private equity CFOs is accentuated by the uncertainty in gauging the fair value of the funds underlying the pool. Even if at the time of the transfer of the underlying PEFs to

the SPV, an independent valuation helps to determine the price of the fund of funds (it is generally carried out by a third party such as a financial auditor), such an assessment is subjective especially for unlisted and privately held assets. Moreover, it is mostly a one-off exercise happening at the inception of the CFO that is not regularly renewed (it is costly).

The objective of this note is to put in place a realistic yet tractable modelling framework that encompasses the uncertainty in the reported NAVs as well as the randomness of the NAV variations, drawdown strategies and repayment policies. In a standard approach, drawdowns and return payments could be treated as investment and consumption processes and the overall valuation of the PEF could be viewed as a problem of utility optimisation. Instead, in the present framework, drawdowns and repayments are considered as totally exogenous random processes and valuing a PEF reduces to solving a stochastic differential equation (SDE).

Effectively, a three-stage model is built: a standard lognormal process is utilised for the dynamics of the investment value combined with a randomly drawn initial value; two squared Bessel processes are utilised for the dynamics of the rates of drawdowns and repayments. Such a model allows fairly easily the calculation of the first and second moments of the drawdown, NAV and repayment distributions. Finally, we show how to build a reasonable model for private equity CFOs (pooled PEFs) and this is illustrated after inferring the model parameters from historical observations.

2. Commitments, Drawdowns, Repayments and Valuation

We give ourselves a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t \geq 0}$. We assume that the space $(\Omega, \mathcal{F})$ is large enough to support all random variables that are introduced hereafter. The filtration itself is generated by the various stochastic processes that are introduced and satisfies the “usual conditions” (see [3]). The probability $\mathbb{P}$ is the natural or historical probability measure and all Brownian motions that are used are defined as standard $\mathbb{P}$ -Brownian motions.

Initial commitments to the fund are supposed to be equal to 1. The fund has a legal maturity T and commitment period T_* , with $T_* \leq T$. Cumulated drawdowns up to time t are denoted D_t and we assume that at any time t , the total drawn amount is no larger than the initial committed amount, i.e. $D_t \leq 1$. The undrawn committed amount up to time t is denoted $\bar{D}_t$ so that we have

$$D_t = 1 - \bar{D}_t, \quad \text{with } D_0 = 0 \quad \text{and} \quad \bar{D}_0 = 1.$$

The drawdowns are carried out between time t and $t + dt$ up to the end of the commitment time T_* at a non-negative rate δ_t from what was left undrawn at time t , that is $\bar{D}_t$, which writes

$$dD_t = \delta_t \bar{D}_t 1_{(t \leq T_*)} dt.$$

Using the preceding assumptions, namely that $dD_t = -d\bar{D}_t$ and $\bar{D}_0 = 1$, we find that

$$\begin{aligned} dD_t &= \delta_t \exp \left(- \int_0^{t \wedge T_*} \delta_s ds \right) 1_{(t \leq T_*)} dt, \\ D_t &= 1 - \exp \left(- \int_0^{t \wedge T_*} \delta_s ds \right). \end{aligned} \quad (2.1)$$

We can check that we indeed have $D_0 = 0$ and $D_t \leq 1$ at any t . Note that this approach allows a certain fraction of the committed amount not to be drawn in finite time given that only for $t = T_* = \infty$ do we have $D_\infty = 1$. Since the maturity of the fund is finite and the commitment period is limited, this might seem slightly peculiar. However, it is not necessarily in contradiction with reality: the manager might keep a certain cash buffer, waiting for investment opportunities and, besides, with a high drawing rate δ_t , the model can still allow asymptotically getting very close to 1 in finite time.

As drawdowns occur, the available cash is immediately invested in managed assets and the portfolio begins to accumulate. Then cash or marketable securities are received as underlying investments are exited and, finally, returns and proceeds can be distributed to the fund partners. We assume that the repayment of returns and proceeds, whose time- t cumulated value is denoted R_t , is distributed between time t and $t + dt$ at a non-negative rate ρ_t . Note that the repayments are calculated on the basis of the time- t invested portfolio value V_t . The idea is that undrawn amounts are not handed back to the partner, save at the closing of the fund, and only the invested drawn amounts can be the subject to repayments.^a Thus, the proceeds dR_t repaid between time t and $t + dt$ and the cumulated returns R_t at time t , which potentially includes the final reimbursement of the assets at the maturity T of the fund, are given by

$$dR_t = \rho_t V_t dt, \quad \text{if } t < T, \quad \text{and} \quad R_t = \int_0^t \rho_s V_s ds + V_t 1_{(t=T)}. \quad (2.2)$$

Note also that, according to the performance of the fund, the model obviously allows the handed-back cumulated repayments to be greater (respectively, smaller) than the drawn-down committed amounts, implying a possibly positive (respectively, certainly negative) IRR over the investment period for the fund partners.

The question is now to model the net asset value V_t of the fund portfolio by taking into consideration the drawdown inflows and the repayment outflows. We first consider that the fund manager follows some investment guidelines, which translate into a certain risk/return target, i.e. a volatility/yield profile. Hence we assume that the return on any cash flow invested in the fund can be described at time t by an average yield μ_t (net of tax and fee) and a non-negative volatility σ_t .

^aThe capital amount is only committed. As long as it is not drawn down into the fund, the LPs can invest it outside the fund. The capital can also be physically pledged to the fund (which uses it as a cash collateral). Yet any undrawn amount is deposited in a safe saving account and the profit is left to the LPs.

We are now equipped for characterising the change in the value of the portfolio dV_t between time t and $t + dt$ that writes

$$dV_t = V_t(\mu_t dt + \sigma_t dW_t^\mu) + dD_t - dR_t. \quad (2.3)$$

The change in the net asset value is made of the performance of the existing investment V_t . It is affected by a potential random factor, W_t^μ , representing a standard Brownian motion. The net asset value is also augmented by the drawdowns and decreased by the repayments. Thanks to the definition of dR_t , this equation can be rewritten

$$dV_t = [dD_t/dt + (\mu_t - \rho_t)V_t]dt + \sigma_t V_t dW_t^\mu, \quad \text{if } t < T, \quad (2.4)$$

which is the SDE of a so-called inverse gamma process (for as long as $t < T_*$, since it turns into a standard lognormal process thereafter, as $dD_t = 0$). We set

$$Z_t = \exp \left(\int_0^t (\mu_s - \rho_s) ds - \frac{1}{2} \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^\mu \right)$$

and the time- t solution to this equation starting from $0 \leq s \leq t$ is given by

$$V_t = \frac{Z_t}{Z_s} V_s + \int_s^t \frac{Z_t}{Z_u} dD_u. \quad (2.5)$$

Note that, a PEF is such that its final value is $V_T = 0$ (because of the final cash flow repayment). Also, the initial value of the fund is $V_0 = 0$ and hence the result could potentially simplify. Nevertheless, we keep this solution as is (starting from $s \geq 0$), because it is practical for old vintages, namely funds with a history of drawdowns and a currently positive asset value as opposed to brand new funds.

To illustrate the model, we will suppose that the fund maturity and commitment period are unlimited, that its starting value $V_0 = 0$ and also that the investment in the assets is non-risky, namely, at any time t , $\sigma_t = 0$. This enables us to construct a fully deterministic fund valuation and to build the corresponding cash flow time series.^b Two examples are presented on the next few charts (see Figs. 1 and 2).

The functions that are used to draw these charts are the result of the two sets of assumptions. They are detailed in Appendix A.3. The resulting patterns can be compared with real data obtained from Venture Economics (see Appendix A.4). This illustrates, in spite of the randomness factor that is not accounted for in the deterministic funds of Appendix A.3, that our models are suitable “in average” for cash flow analysis, especially the one that makes the drawdowns and repayments linear functions of the time (see Fig. 2).

So far, the only random element in the NAV dynamics is the one affecting the investment return via the volatility term $\sigma_t dW_t^\mu$. To account for the inaccurate valuation of the fund at a given time, we assume that the initial value of Eq. (2.5), V_s , instead of being drawn out of a Dirac distribution, as is most usual when solving

^bThroughout this document, whenever $dD(t)$ and $dR(t)$ appear on the chart legends, they have to be understood instead as shortcuts for $dD(t)/dt\Delta t$ and $dR(t)/dt\Delta t$, where $\Delta t = 0.25$ is the time step that is used to construct the graphs.

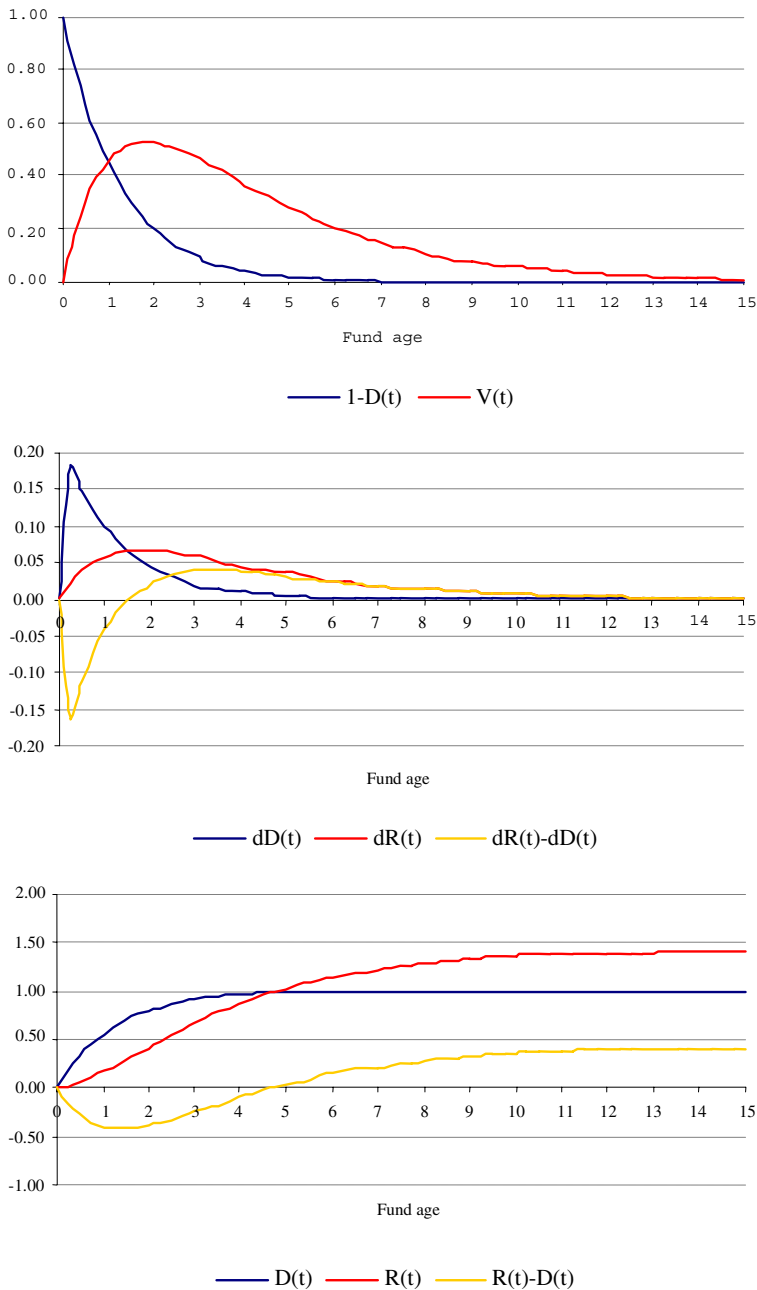


Fig. 1. Deterministic commitments, drawdowns, repayments and fund value. The volatility σ_t is zero and all rates δ_t , μ_t and ρ_t are constant equal to δ , μ and ρ respectively where $\delta = 0.80$, $\mu = 0.10$, $\rho = 0.15$.

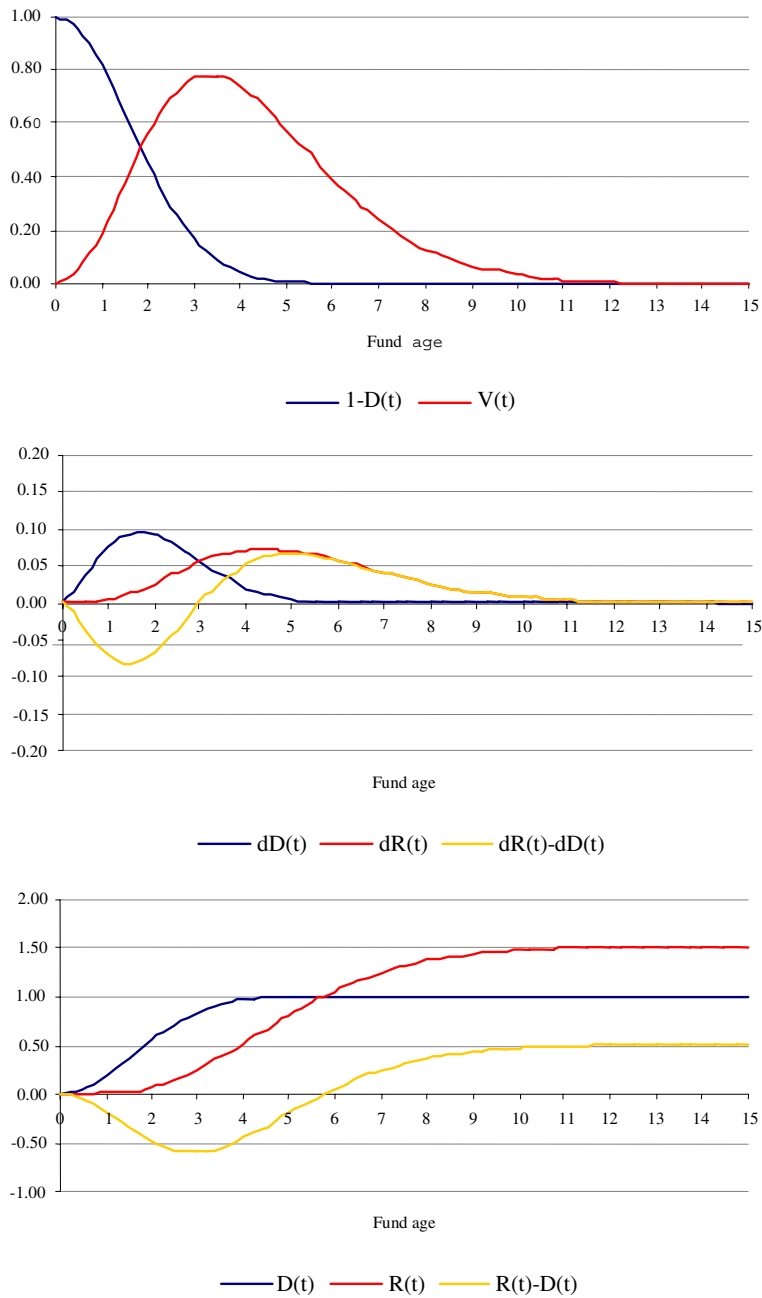


Fig. 2. Deterministic commitments, drawdowns, repayments and fund value. The volatility σ_t is zero, the yield μ_t is constant equal to μ whereas the drawdown δ_t and the repayment ρ_t are linear functions of the time, i.e. $\delta_t = \delta t$ with $\delta > 0$ and $\rho_t = \rho t$ with $\rho > 0$ where $\delta = 0.40$, $\mu = 0.10$, $\rho = 0.15$.

an SDE, is actually drawn out of a lognormal distribution with parameters $(\ln \bar{V}_s - \varepsilon^2/2, \varepsilon^2)$. We assume moreover that the valuation error is independent from W_t^μ . Therefore, we have

$$V_s = \bar{V}_s e^{-\frac{\varepsilon^2}{2} + \varepsilon X}, \quad (2.6)$$

where the estimation error X is a standard normal distribution independent from W_t^μ so that the estimation of the fund value $\bar{V}_s$ provided by the financial auditor conditional on $\mathcal{F}_s$ is actually unbiased. The presence of an error in the assessment of the NAV at a given time s will obviously impact the calculation of any incomplete expectation and of all higher moments of the future fund value V_t , $s \leq t$.

We have also seen that the drawdown rate and the repayment rate are essential components of the behaviour of the fund performance. In a purely deterministic world, we have examined two examples, the second one being the most promising in its ability to reproduce the usual average pattern and shape of drawdown and repayment policies that are generally observed. In the first example, the drawdown and repayment rates are constant, which implicitly gives to drawdowns and repayments the shape of an exponential distribution function. In the second example, the drawdown and repayment rates are linearly increasing with time, which implicitly gives to drawdowns and repayments the shape of a Weibull distribution function, with a tail exponent equal to 2. It is this sort of shape that we want to keep, yet with the addition of a stochastic component that will allow a certain degree of irregularity in the cash inflows and outflows. The second constraint that needs to be imposed on a drawdown or repayment model is to keep those rates, respectively δ_t or ρ_t , non-negative at any time (or at least to have a very small probability of their becoming negative), so that $\int_0^t \delta_s ds$ and $\int_0^t \rho_s ds$ are non-negative. These elements led to consider the following dynamics as very appropriate stochastic models for the problem at stake:

$$\begin{aligned} d\delta_t &= (c_1 + c_2\delta_t)dt + c_3\sqrt{\delta_t}dW_t^\delta, \\ d\rho_t &= (q_1 + q_2\rho_t)dt + q_3\sqrt{\rho_t}dW_t^\rho, \end{aligned} \quad (2.7)$$

with $c_1, c_3, q_1, q_3 > 0$. These SDEs are standard squared Bessel processes. Cox *et al.* [12, 13] first introduced the squared Bessel process in finance to model interest rates,^c where it is now generally known as the CIR or square-root process. The existence and uniqueness of a strong solution to the SDE of such a squared Bessel process for every initial value is a well-established fact. Moreover, if we denote $\tau_0^\delta = \inf\{t \geq 0; \delta_t = 0\}$ and $\tau_0^\rho = \inf\{t \geq 0; \rho_t = 0\}$, the first time δ_t and ρ_t reach zero, it is possible to establish that if $c_1 > c_3^2/2$ and $q_1 > q_3^2/2$, then $\tau_0^\delta = \infty$ and $\tau_0^\rho = \infty$ almost surely. For a proof, see for example [4] or [5]. Thus, for these

^cCox *et al.* [13] more exactly modelled the short-term interest rate r_t as $dr_t = a(b - r_t)dt + c\sqrt{r_t}dW_t$, with $a, b, c > 0$. The structure of the drift $a(b - r_t)$ creates a reversion of the process towards the level b , which is not the effect that is necessarily sought after here. In particular, the drawdown and repayment processes are not assumed to be ergodic.

parameter restrictions, we are almost sure that these processes will remain positive. For further properties of squared Bessel processes, the interested reader can refer to [4], [6] or [7]. A general presentation of CIR processes in finance is provided by Duffie *et al.* [8]. For extensions of the squared Bessel process to the case of time-varying parameters, some mathematical aspects are presented in [9] and applications to interest rate modelling are discussed in [10] or [11]. These references may provide future researchers with some ways to enhance the present equity fund valuation model. In the current approach, we opted for time-independent parameters to keep it statistically simpler.

3. First-order and Second-order Moments

As such, the model now allows calculating at any time t the expected cumulative and instantaneous drawdowns, the expected value of the equity fund and the expected instantaneous returns. Let $\mathbb{E}[\cdot|\mathcal{F}_t]$ denote the expectation operator under the natural probability measure $\mathbb{P}$ conditional on the information $\mathcal{F}_t$ revealed up to time t . The following result is given in many financial textbooks, including in the articles of Cox *et al.* [12, 13], and is a consequence of the Feynman–Kac formula (see e.g. [14] or [3]). For $s \leq t$,

$$\mathbb{E} \left[\exp \left(- \int_s^t \delta_u du \right) | \mathcal{F}_s \right] = \exp(-M(t-s) - N(t-s)\delta_s). \quad (3.1)$$

In particular, we have

$$\mathbb{E}[D_t|\mathcal{F}_s] = 1 - \bar{D}_s \exp(-M(t-s) - N(t-s)\delta_s), \text{ if } t \leq T_*, \quad (3.2)$$

with $M(t) = F_{0,1}(t)$ and $N(t) = G_{0,1}(t)$, where the functions $F_{\lambda,\mu}$ and $G_{\lambda,\mu}$ are defined^d in Appendix B.2.

The expected instantaneous drawdowns are obtained by deriving $\mathbb{E}[\bar{D}_t \exp(-\lambda \delta_t)|\mathcal{F}_s]$ with respect to λ and taking the resulting function at $\lambda = 0$, as described in Eq. (B.7) and the remainder of Appendix B.2. Note that this actually provides the same result as deriving $\mathbb{E}[D_t|F_s]$ with respect to t since it turns out that

$$\mathbb{E}[dD_t/dt|\mathcal{F}_s] = -\frac{d}{d\lambda} \mathbb{E}[\bar{D}_t \exp(-\lambda \delta_t)|\mathcal{F}_s]|_{\lambda=0} = \frac{d}{dt} \mathbb{E}[D_t|\mathcal{F}_s]. \quad (3.3)$$

This finally means that

$$\mathbb{E}[dD_t/dt|\mathcal{F}_s] = \bar{D}_s(\dot{M}(t-s) + \dot{N}(t-s)\delta_s) \exp(-M(t-s) - N(t-s)\delta_s). \quad (3.4)$$

It is easy to check that $\dot{M}(t) = \partial M(t)/\partial t$ and $\dot{N}(t) = \partial N(t)/\partial t$ are indeed respectively equal to $F'_{0,1}(t) = \partial F_{\lambda,\mu}(t)/\partial \lambda|_{\lambda=0,\mu=1}$ and $G'_{0,1}(t) = \partial G_{\lambda,\mu}(t)/\partial \lambda|_{\lambda=0,\mu=1}$.

The next step is the calculation of the expected NAV V_t . A natural and simplifying assumption is to postulate also that the three Brownian motions W_t^μ , W_t^δ

^dObviously, throughout this section, the functions $F_{\lambda,\mu}$ and $G_{\lambda,\mu}$ use either the parameters c_1, c_2, c_3 or the parameters q_1, q_2, q_3 depending on whether we manipulate the process δ_t or ρ_t . It should be clear given the context though.

and W_t^ρ are independent. From Eq. (2.5) and assuming that the pricing error X is independent from W_t^ρ too, the expected NAV can be written

$$\mathbb{E}[V_t|\mathcal{F}_s] = \bar{V}_s \mathbb{E}[Z_t/Z_s|\mathcal{F}_s] + \int_s^t \mathbb{E}[dD_u|\mathcal{F}_s] \mathbb{E}[Z_t/Z_u|\mathcal{F}_s]. \quad (3.5)$$

Applying the Feynman–Kac formula, the value of $\mathbb{E}[Z_t/Z_s|\mathcal{F}_s]$ becomes

$$\mathbb{E}[Z_t/Z_s|\mathcal{F}_s] = \exp\left(\int_s^t \mu_u du\right) \exp(-M(t-s) - N(t-s)\rho_s). \quad (3.6)$$

We also have

$$\begin{aligned} \mathbb{E}[Z_t/Z_u|\mathcal{F}_s] &= \exp\left(\int_u^t \mu_v dv\right) \left[\frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1]N(t-u) + 1\right]^{-2q_1/q_3^2} \\ &\quad \times \exp\left(-M(t-u) - \frac{e^{q_2(u-s)}N(t-u)\rho_s}{\frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1]N(t-u) + 1}\right). \end{aligned} \quad (3.7)$$

Note that Eq. (3.7) required the expression of $\mathbb{E}[\exp(-\int_u^t \rho_v dv)|\mathcal{F}_s]$, which is obtained by conditioning first on $\mathcal{F}_u$ with $s \leq u$, which, thanks to the result of Eq. (3.6), boils down to calculating $\exp(-M(t-u))$ times the Laplace transform of ρ_u with a Laplace coefficient equal to $N(t-u)$. The expression for the Laplace transform of a squared Bessel process is given e.g. in [5] and recalled in Appendix B.1. In order to complete the calculation of $\int_s^t \mathbb{E}[dD_u Z_t/Z_u|\mathcal{F}_s]$ and hence of $\mathbb{E}[V_t|\mathcal{F}_s]$, we would have to resort to some sort of quadrature approximation or other type of numerical integration technique.

To determine the expected instantaneous repayment cash flows, we have to calculate $\mathbb{E}[dR_t/dt|\mathcal{F}_s] = \mathbb{E}[\rho_t V_t|\mathcal{F}_s]$, which in turn, from Eq. (3.5), requires the computation of $\mathbb{E}[\rho_t Z_t/Z_s|\mathcal{F}_s]$ and $\mathbb{E}[\rho_t Z_t/Z_u|\mathcal{F}_s]$. Using the same argument as for Eq. (3.4), we have

$$\begin{aligned} \mathbb{E}[\rho_t Z_t/Z_s|\mathcal{F}_s] &= \exp\left(\int_s^t \mu_u du\right) (\dot{M}(t-s) + \dot{N}(t-s)\rho_s) \\ &\quad \times \exp(-M(t-s) - N(t-s)\rho_s). \end{aligned} \quad (3.8)$$

The problematic of the calculation of the expectation of $\rho_t Z_t/Z_u$ conditional on $\mathcal{F}_s$ is finally not very different, since it implies the valuation of $d\mathbb{E}[\exp(-\int_u^t \rho_v dv)|\mathcal{F}_s]/dt$ or, equivalently, of $d\mathbb{E}[\exp(-\lambda\rho_t - \int_u^t \rho_v dv)|\mathcal{F}_s]/d\lambda|_{\lambda=0}$, which is calculated thanks to Eq. (3.7) as

$$\begin{aligned} &\mathbb{E}\left[\rho_t \exp\left(-\int_u^t \rho_v dv\right) | \mathcal{F}_s\right] \\ &= \left[\dot{M}(t-u) + \frac{\dot{N}(t-u)}{\frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1]N(t-u) + 1}\right] \end{aligned}$$

$$\begin{aligned} & \times \left(\frac{q_1}{q_2} [e^{q_2(u-s)} - 1] + \frac{e^{q_2(u-s)} \rho_s}{\frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1] N(t-u) + 1} \right) \Bigg] \\ & \times \left[\frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1] N(t-u) + 1 \right]^{-2q_1/q_3^2} \\ & \times \exp \left(-M(t-u) - \frac{e^{q_2(u-s)} N(t-u) \rho_s}{\frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1] N(t-u) + 1} \right). \end{aligned} \quad (3.9)$$

Again, to finalise the estimation of $\mathbb{E}[dR_t/dt|\mathcal{F}_s]$, we would have to use a numerical integration technique.

Finding the higher moments of the different variables is slightly more tedious. Still the second-order moment is an interesting measure since it allows those who are used to benchmarking their risk in terms of standard deviation to calculate a risk-adjusted return on their deal. For example, this measure can help the swap provider to scale his profit buffer. As far as the cumulative and instantaneous drawdowns are concerned, we immediately have for $t \leq T_*$

$$\mathbb{E}[\bar{D}_t^2|\mathcal{F}_s] = \bar{D}_s^2 \exp(-P(t-s) - Q(t-s)\delta_s), \quad (3.10)$$

where $P(t) = F_{0,2}(t)$ and $Q(t) = G_{0,2}(t)$. The second moment of the instantaneous drawdowns is obtained by deriving $\mathbb{E}[\bar{D}_t^2 \exp(-\lambda \delta_t)|\mathcal{F}_s]$ twice with respect to λ and taking the resulting function at $\lambda = 0$, as described in Appendix B.2. Thus

$$\begin{aligned} \mathbb{E}[(dD_t/dt)^2|\mathcal{F}_s] &= \bar{D}_s^2 [-(F''_{0,2}(t-s) + G''_{0,2}(t-s)\delta_s) \\ &\quad + (F'_{0,2}(t-s) + G'_{0,2}(t-s)\delta_s)^2] \\ &\quad \times \exp(-P(t-s) - Q(t-s)\delta_s), \end{aligned} \quad (3.11)$$

where $F''_{\lambda,\mu}(t) = \partial^2 F_{\lambda,\mu}(t)/\partial \lambda^2$ and $G''_{\lambda,\mu}(t) = \partial^2 G_{\lambda,\mu}(t)/\partial \lambda^2$. As for the second order moment of the fund value, we need to calculate

$$\begin{aligned} \mathbb{E}[V_t^2|\mathcal{F}_s] &= \bar{V}_s^2 e^{\varepsilon^2} \mathbb{E}[(Z_t/Z_s)^2|\mathcal{F}_s] + 2\bar{V}_s \int_s^t \mathbb{E}[(Z_t/Z_s)(Z_t/Z_v)|\mathcal{F}_s] \mathbb{E}[dD_v|\mathcal{F}_s] \\ &\quad + \int_s^t \int_s^t \mathbb{E}[(Z_t/Z_u)(Z_t/Z_v)|\mathcal{F}_s] \mathbb{E}[dD_u dD_v|\mathcal{F}_s]. \end{aligned} \quad (3.12)$$

First, we have

$$\begin{aligned} \mathbb{E}[(Z_t/Z_s)^2|\mathcal{F}_s] &= \exp \left(2 \int_s^t \mu_u du + \int_s^t \sigma_u^2 du \right) \\ &\quad \times \exp(-P(t-s) - Q(t-s)\rho_s). \end{aligned} \quad (3.13)$$

Then, for $s \leq v \leq t$, we can write

$$\begin{aligned} \mathbb{E} \left[\frac{Z_t}{Z_s} \frac{Z_t}{Z_v} | \mathcal{F}_s \right] &= \exp \left(\int_s^v \mu_{v'} dv' + 2 \int_v^t \mu_{v'} dv' + \int_v^t \sigma_{v'}^2 dv' \right) \\ &\times \exp(-P(t-v) - F_{Q(t-v),1}(v-s) \\ &- G_{Q(t-v),1}(v-s)\rho_s). \end{aligned} \quad (3.14)$$

Equation (3.14) is obtained first by conditioning on $\mathcal{F}_v$ and then by unfolding the main result of Appendix B.2. Also, assuming that $u \leq v$, we can write

$$\begin{aligned} \mathbb{E} \left[\frac{Z_t}{Z_u} \frac{Z_t}{Z_v} | \mathcal{F}_s \right] &= \exp \left(\int_u^v \mu_{v'} dv' + 2 \int_v^t \mu_{v'} dv' + \int_v^t \sigma_{v'}^2 dv' - P(t-v) - F_{Q(t-v),1}(v-u) \right) \\ &\times \left[\frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1] G_{Q(t-v),1}(v-u) + 1 \right]^{-2q_1/q_3^2} \\ &\times \exp \left(\frac{e^{q_2(u-s)} G_{Q(t-v),1}(v-u)\rho_s}{\frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1] G_{Q(t-v),1}(v-u) + 1} \right). \end{aligned} \quad (3.15)$$

Eq. (3.15) is obtained first by conditioning on $\mathcal{F}_u$ then by conditioning on $\mathcal{F}_v$ and first by using the main result of Appendix B.2 and eventually by applying the main result of Appendix B.1. One can check that (3.13) and (3.14) are particular cases of (3.15).

Finally, for the last important expectation of Eq. (3.12), after writing

$$\mathbb{E} \left[\frac{d\bar{D}_u}{du} \frac{d\bar{D}_v}{dv} | \mathcal{F}_s \right] = \frac{d^2}{d\lambda_1 d\lambda_2} \mathbb{E} \left[\bar{D}_u^2 \exp(-\lambda_1 \delta_u) \mathbb{E} \left[\frac{\bar{D}_v}{\bar{D}_u} \exp(-\lambda_2 \delta_v) | \mathcal{F}_u \right] | \mathcal{F}_s \right] \Big|_{\lambda_1=\lambda_2=0},$$

for $u \leq v$, we find that

$$\begin{aligned} \mathbb{E}[dD_u dD_v | \mathcal{F}_s] &= dudv \bar{D}_s^2 [-G'_{0,1}(v-u)(F''_{N(v-u),2}(u-s) + G''_{N(v-u),2}(u-s)\delta_u) \\ &- (F_{0,1}(v-u) + G'_{0,1}(v-u)(F'_{N(v-u),2}(u-s) \\ &+ G'_{N(v-u),2}(u-s)\delta_u)) \times (F'_{0,1}(v-u) + F'_{N(v-u),2}(u-s) \\ &+ G'_{N(v-u),2}(u-s)\delta_u)] \times \exp(-F_{0,1}(v-u) - F_{N(v-u),2}(u-s) \\ &- G_{N(v-u),2}(u-s)\delta_u). \end{aligned} \quad (3.16)$$

It is easy to check that, whenever $u = v = t$, since $F_{\lambda,\mu}(0) = F'_{\lambda,\mu}(0) = F''_{\lambda,\mu}(0) = 0$, $G_{\lambda,\mu}(0) = \lambda$, $G'_{\lambda,\mu}(0) = 1$ and $G''_{\lambda,\mu}(0) = 0$, Eq. (3.16) reduces to (3.11). Putting everything together and using some numerical integration technique, the calculation of $\mathbb{E}[V_t^2 | \mathcal{F}_s]$ can be completed.

As far as the second order moment of the instantaneous repayments is concerned, we have to calculate

$$\begin{aligned}\mathbb{E}[\rho_t^2 V_t^2 | \mathcal{F}_s] &= \bar{V}_s^2 e^{\varepsilon^2} \mathbb{E}[\rho_t^2 (Z_t/Z_s)^2 | \mathcal{F}_s] \\ &\quad + 2\bar{V}_s \int_s^t \mathbb{E}[\rho_t^2 (Z_t/Z_s)(Z_t/Z_v) | \mathcal{F}_s] \mathbb{E}[dD_v | \mathcal{F}_s] \\ &\quad + \int_s^t \int_s^t \mathbb{E}[\rho_t^2 (Z_t/Z_u)(Z_t/Z_v) | \mathcal{F}_s] \mathbb{E}[dD_u dD_v | \mathcal{F}_s].\end{aligned}\quad (3.17)$$

Using the approach already applied to Eq. (3.11), we have by derivation

$$\begin{aligned}\mathbb{E}\left[\rho_t^2 \left(\frac{Z_t}{Z_s}\right)^2 \middle| \mathcal{F}_s\right] &= \exp\left(2 \int_s^t \mu_{v'} dv' + \int_s^t \sigma_{v'}^2 dv'\right) [- (F_{0,2}''(t-s) + G_{0,2}''(t-s)\rho_s) \\ &\quad + (F_{0,2}'(t-s) + G_{0,2}'(t-s)\rho_s)^2] \\ &\quad \times \exp(-P(t-s) - Q(t-s)\rho_s).\end{aligned}\quad (3.18)$$

The expectation of $\rho_t^2 (Z_t/Z_u)(Z_t/Z_s)$ conditional on $\mathcal{F}_s$ can be computed by noticing that

$$\mathbb{E}\left[\rho_t^2 \frac{Z_t}{Z_s} \frac{Z_t}{Z_v} \middle| \mathcal{F}_s\right] = \frac{d^2}{d\lambda^2} \mathbb{E}\left[\frac{Z_v}{Z_s} \mathbb{E}\left[\left(\frac{Z_t}{Z_v}\right)^2 \exp(-\lambda \rho_t) \middle| \mathcal{F}_v\right] \middle| \mathcal{F}_s\right] \bigg|_{\lambda=0},$$

which yields

$$\begin{aligned}\mathbb{E}\left[\rho_t^2 \frac{Z_t}{Z_s} \frac{Z_t}{Z_v} \middle| \mathcal{F}_s\right] &= \exp\left(\int_s^v \mu_{v'} dv' + 2 \int_v^t \mu_{v'} dv' + \int_v^t \sigma_{v'}^2 dv'\right) \\ &\quad \times \left[-F_{0,2}''(t-v) - G_{0,2}''(t-v)(F_{Q(t-v),1}'(v-s) + G_{Q(t-v),1}'(v-s)\rho_s) \right. \\ &\quad \left.- G_{0,2}'^2(t-v)(F_{Q(t-v),1}''(v-s) + G_{Q(t-v),1}''(v-s)\rho_s) \right. \\ &\quad \left.+ (F_{0,2}'(t-v) + G_{0,2}'(t-v)(F_{Q(t-v),1}'(v-s) + G_{Q(t-v),1}'(v-s)\rho_s))^2\right] \\ &\quad \times \exp(-P(t-v) - F_{Q(t-v),1}(v-s) - G_{Q(t-v),1}(v-s)\rho_s).\end{aligned}\quad (3.19)$$

Likewise, assuming that $u \leq v$, the expectation of $\rho_t^2 (Z_t/Z_u)(Z_t/Z_v)$ conditional on $\mathcal{F}_s$ can be written

$$\mathbb{E}\left[\rho_t^2 \frac{Z_t}{Z_u} \frac{Z_t}{Z_v} \middle| \mathcal{F}_s\right] = \frac{d^2}{d\lambda^2} \mathbb{E}\left[\mathbb{E}\left[\frac{Z_v}{Z_u} \mathbb{E}\left[\left(\frac{Z_t}{Z_v}\right)^2 \exp(-\lambda \rho_t) \middle| \mathcal{F}_v\right] \middle| \mathcal{F}_u\right] \middle| \mathcal{F}_s\right] \bigg|_{\lambda=0}.$$

After some lengthy calculations, we obtain

$$\begin{aligned}
 \mathbb{E} \left[\rho_t^2 \frac{Z_t}{Z_u} \frac{Z_t}{Z_v} \middle| \mathcal{F}_s \right] &= \exp \left(\int_u^v \mu_{v'} dv' + 2 \int_v^t \mu_{v'} dv' + \int_v^t \sigma_{v'}^2 dv' \right) \\
 &\quad \times [-(F''_{0,2}(t-v) + G''_{0,2}(t-v)F'_{Q(t-v),1}(v-u) \\
 &\quad + G'^2_{0,2}(t-v)F''_{Q(t-v),1}(v-u) + C(u-s, v-u)) \\
 &\quad + (F'_{0,2}(t-v) + G'_{0,2}(t-v)F'_{Q(t-v),1}(v-u) \\
 &\quad + B(u-s, v-u))^2] \times \exp(-P(t-v) - F_{Q(t-v),1}(v-u)) \\
 &\quad \times A^{-\frac{2q_1}{q_3}}(u-s) \exp \left(-\frac{e^{q_2(u-s)} G_{Q(t-v),1}(v-u) \rho_s}{A(u-s)} \right) \quad (3.20)
 \end{aligned}$$

where the functions $A(u-s)$, $B(u-s, v-u)$ and $C(u-s, v-u)$ are defined as

$$\begin{aligned}
 A(u-s) &= \frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1] G_{Q(t-v),1}(v-u) + 1 \\
 B(u-s, v-u) &= \frac{G'_{0,2}(t-v) G'_{Q(t-v),1}(v-u)}{A(u-s)} \left(\frac{q_1}{q_2} [e^{q_2(u-s)} - 1] + \frac{e^{q_2(u-s)} \rho_s}{A(u-s)} \right) \\
 C(u-s, v-u) &= \frac{1}{A(u-s)} \left[-\frac{q_3^2}{2q_2} [e^{q_2(u-s)} - 1] G'_{0,2}(t-v) G'_{Q(t-v),1}(v-u) \right. \\
 &\quad \times \left(B(u-s, v-u) + \frac{e^{q_2(u-s)}}{A^2(u-s)} G'_{0,2}(t-v) G'_{Q(t-v),1}(v-u) \rho_s \right) \\
 &\quad + (G''_{0,2}(t-v) G'_{Q(t-v),1}(v-u) + G'^2_{0,2}(t-v) G''_{Q(t-v),1}(v-u)) \\
 &\quad \times \left. \left(\frac{q_1}{q_2} [e^{q_2(u-s)} - 1] + \frac{e^{q_2(u-s)}}{A(u-s)} \rho_s \right) \right].
 \end{aligned}$$

Noting that

$$\begin{aligned}
 A(0) &= 1, \\
 B(0, v-u) &= G'_{0,2}(t-v) G'_{Q(t-v),1}(v-u) \rho_s \\
 B(0, 0) &= G'_{0,2}(t-v) \rho_s, \\
 C(0, v-u) &= (G''_{0,2}(t-v) G'_{Q(t-v),1}(v-u) + G'^2_{0,2}(t-v) G''_{Q(t-v),1}(v-u)) \rho_s, \\
 C(0, 0) &= G''_{0,2}(t-v) \rho_s,
 \end{aligned}$$

it is possible to check that (3.20) can simplify into (3.19), when $u = s$, and into (3.18), when $v = u = s$, as expected.

The results obtained so far are sufficient for the estimation of the expected future cash flows produced by some PEF. In particular, a private equity CFO often issues note tranches indexed to LIBOR rates by swapping with a third party the cash flows

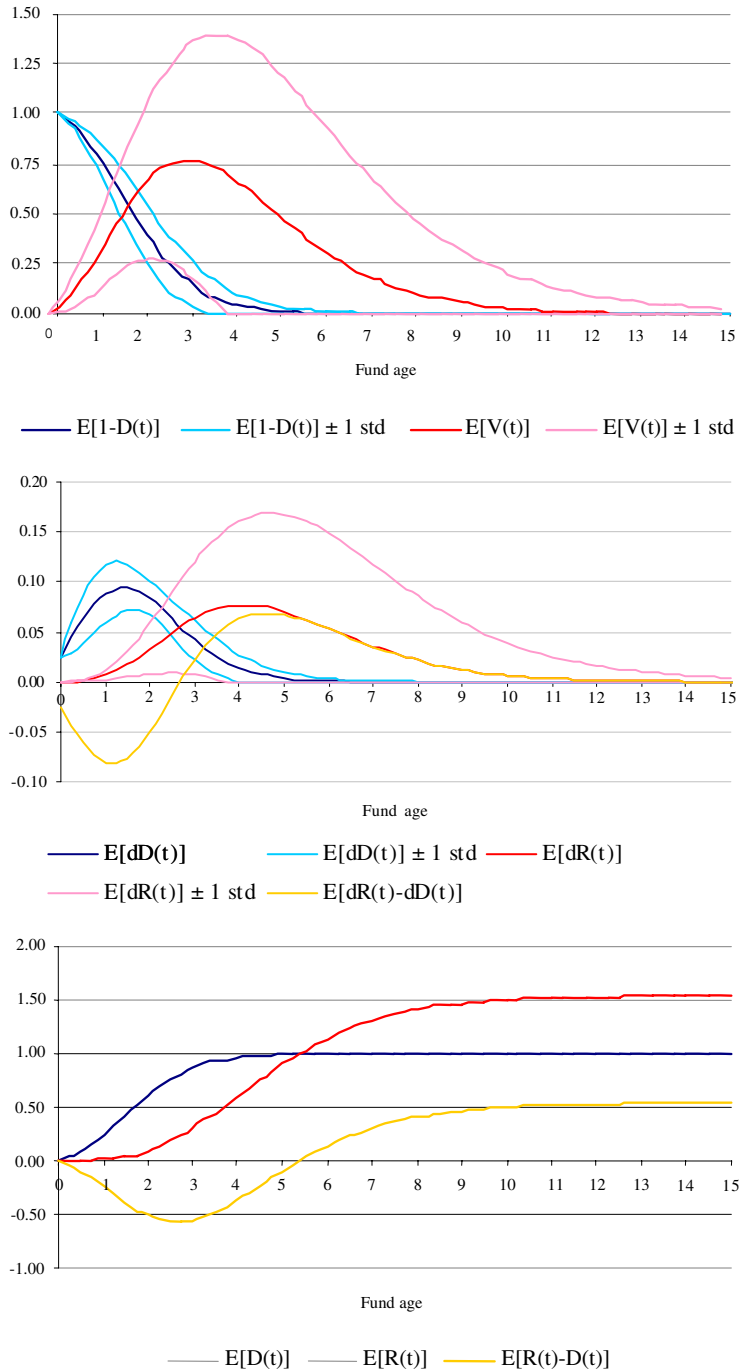


Fig. 3. Expected commitments, drawdowns, repayments and fund value. $\delta = 0.10$, $c_1 = 0.30$, $c_2 = 0.30$, $c_3 = 0.40$, $\rho = 0.00$, $q_1 = 0.10$, $q_2 = 0.10$, $q_3 = 0.20$, $\mu = 0.15$, $\sigma = 0.25$.

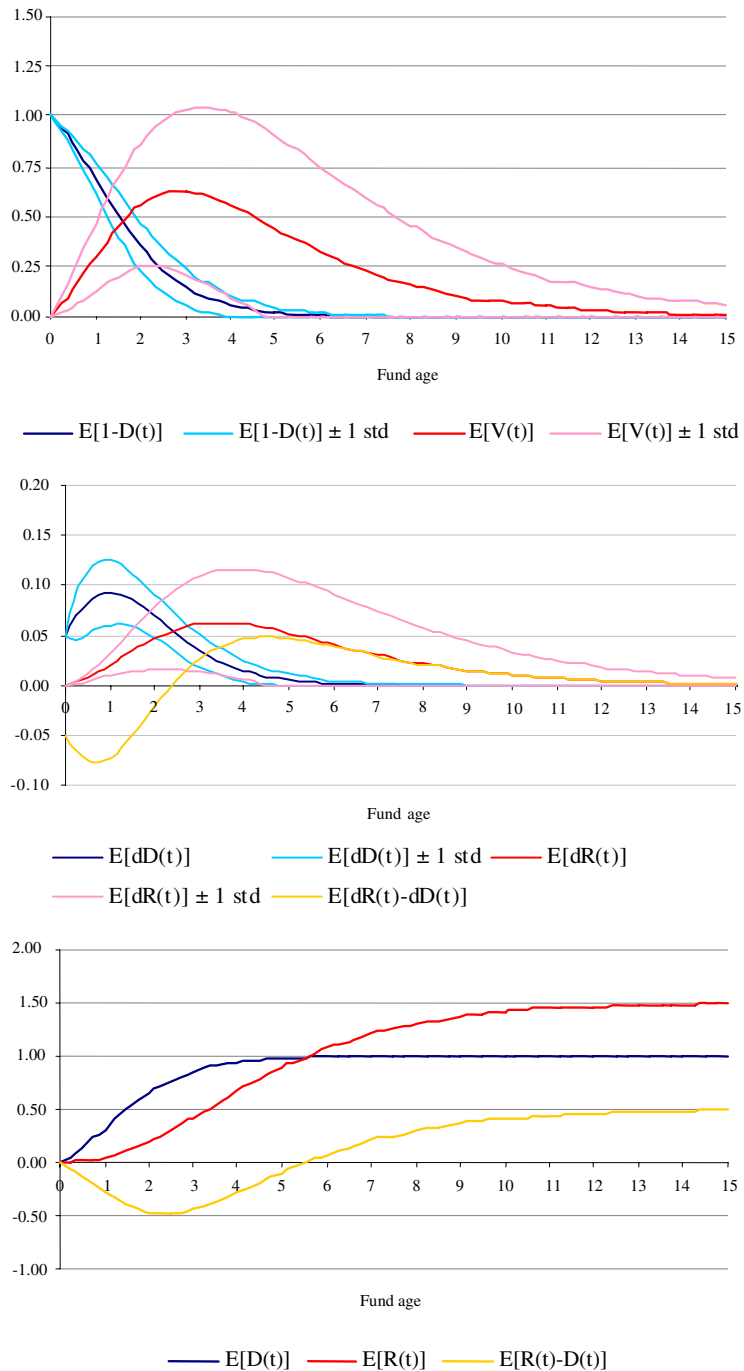


Fig. 4. Expected commitments, drawdowns, repayments and fund value. $\delta = 0.20$, $c_1 = 0.40$, $c_2 = -0.10$, $c_3 = 0.40$, $\rho = 0.20$, $q_1 = 0.10$, $q_2 = -0.10$, $q_3 = 0.15$, $\mu = 0.15$, $\sigma = 0.25$.

of the underlying PEFs against the required LIBOR rates. For the swap provider, it is of course absolutely essential to know how to price this and the model allows doing just that. In general though, the pricing of PEF-linked products is carried out under the risk-adjusted probability measure $\mathbb{Q}$. Under this probability measure, private equity returns are equal to the risk-free rate r_t instead of μ_t . The likelihood process L_t that allows changing the measure $\mathbb{P}$ into $\mathbb{Q}$ can be defined as

$$L_t = \exp \left(-\frac{1}{2} \int_0^t \left(\frac{\mu_u - r_u}{\sigma_u} \right)^2 du + \int_0^t \frac{\mu_u - r_u}{\sigma_u} dW_u^\mu \right). \quad (3.21)$$

The assumption that the three Brownian motions W_t^μ , W_t^δ and W_t^ρ are independent means that the drifts of the processes δ_t and ρ_t are not affected by the change of probability measure. In practice, it means that we are able to use the historical or statistical values of the diffusion parameters without any correction.

To illustrate some of the model properties, Figs. 3 and 4 present the expected cash flows and their standard deviations obtained from different parameter values of the diffusion processes.

Nevertheless, the most important aspect is actually the fact that we now have a framework that can be utilised by any Monte Carlo engine for simulating sample paths and assigning a price or a risk to any PEF-linked transaction.

4. A Simple Model for Private Equity Collateralised Fund Obligations

The question is now to find a reasonable and parsimonious yet realistic way of modelling a fund of (potentially numerous) PEFs. We also have to bear in mind that in order to estimate the distribution parameters of the individual PEFs only historical data and no implied market data are available. Besides, since, by definition, a fund is doomed because of its limited lifetime, the only way to fit the parameters of the fund value, drawdown and repayment processes is by using data that were observed for past funds of a similar type.

This section can be viewed as a first step for building a Monte Carlo engine that will be a very useful tool for pricing and analysing some of the complex transactions that can be linked to a fund of funds: for example, for estimating the probability that the SPV that issued the bonds will be able to meet its interest and principal payment obligations, for measuring the cost for the SPV of a liquidity facility or for calculating the probability and the cost for the SPV of the cash flow swap provider's defaulting.

Suppose that the fund of funds comprises n individual PEFs. Each of the various funds i can be first characterised by a set of three dates: its investment period spot date $T_\bullet^{(i)}$, its investment period expiration date $T_*^{(i)}$ and its liquidation or termination date $T^{(i)}$, where the following obviously holds: $T_\bullet^{(i)} \leq T_*^{(i)} \leq T^{(i)}$. A fund of PEFs is made of funds that are all started and we are only interested in a particular fund as long as it has not matured. Thus the current date t is such that $T_\bullet^{(i)} \leq t \leq T^{(i)}$. Now depending on the fund t can be larger or smaller than $T_*^{(i)}$.

For each fund, we had $V_{T^{(i)}}^{(i)} = 0$ and $\bar{D}_{T^{(i)}}^{(i)} = 1$ and we observe, though imperfectly, a current value $V_t^{(i)}$ (that goes to zero as soon as $t \geq T^{(i)}$). The total value of the fund of funds is given by $V_t = \sum_{1 \leq i \leq n} w^{(i)} V_t^{(i)}$, where the weights $(w^{(i)})_{1 \leq i \leq n}$ are set and decided at the constitution of the fund of funds and remain unchanged thereafter. We are dealing here with a so-called “static” fund of funds, which is the common set-up for private equity CFOs. The fund of funds matures as soon as $t \geq \max\{i; T^{(i)}\}$. A complete model would necessitate specifying the following system of SDEs for all $1 \leq i \leq n$:

$$\begin{aligned} dV_t^{(i)} &= [dD_t^{(i)}/dt + (\mu_t^{(i)} - \rho_t^{(i)})V_t^{(i)}]dt + \sigma_t^{(i)}V_t^{(i)} \sum_{j=1}^n \alpha_t^{(i,j)} dW_t^{\mu(j)}, \\ dD_t^{(i)} &= \delta_t^{(i)}(1 - D_t^{(i)})1_{(t \leq T^{(i)})} dt, \\ d\delta_t^{(i)} &= (c_1^{(i)} + c_2^{(i)}\delta_t^{(i)})dt + c_3^{(i)}\sqrt{\delta_t^{(i)}} \sum_{j=1}^n \beta_t^{(i,j)} dW_t^{\delta(j)}, \\ d\rho_t^{(i)} &= (q_1^{(i)} + q_2^{(i)}\rho_t^{(i)})dt + q_3^{(i)}\sqrt{\rho_t^{(i)}} \sum_{j=1}^n \gamma_t^{(i,j)} dW_t^{\rho(j)}, \end{aligned} \quad (4.1)$$

where $(W_t^{\mu(j)})_{1 \leq j \leq n}$, $(W_t^{\delta(j)})_{1 \leq j \leq n}$ and $(W_t^{\rho(j)})_{1 \leq j \leq n}$ are three sets of independent Brownian motions and where, for all $1 \leq i \leq n$

$$\sum_{1 \leq j \leq n} \alpha_t^{(i,j)^2} = \sum_{1 \leq j \leq n} \beta_t^{(i,j)^2} = \sum_{1 \leq j \leq n} \gamma_t^{(i,j)^2} = 1,$$

these parameters respectively being the correlation structure between the fund performances, the drawdown decisions and the repayment policies. Obviously, so many parameters would be in practice extremely difficult to fit. Nevertheless, a judicious set of hypotheses can be made to simplify this setting.

A fund can be categorised according to its type of sector, strategy, industry etc. We assume that we have defined m types for which a consistent set of observations and statistics can be derived and that $m \leq n$. Thus we can play on this lower dimension to reduce the number of parameters to estimate. Say that we deal with fund i , $1 \leq i \leq n$, that is of type k , $1 \leq k \leq m$. We can make the following five additional (arguable but reasonable) hypotheses:

- (i) Drawdowns and repayments of a given fund type are subject to the same single systemic random variable together with an idiosyncratic component. However, funds of different types have independent drawdown and repayment policies.
- (ii) Drawdowns and repayments of the same type of fund have the same distribution parameters in so far as they have the same commitment period or lifetime. To account for different commitment periods $T_*^{(i)} - T_{\bullet}^{(i)}$ or lifetimes $T^{(i)} - T_{\bullet}^{(i)}$, the parameters are simply rescaled.
- (iii) Performances of a given fund type are subject to the same multiple systemic random variables together with an idiosyncratic component. Yet, funds of different types do not necessarily have independent performance distributions.

- (iv) For a given fund type, the distribution parameters of the fund performance are identical.
- (v) The initial valuation errors of the various funds are independent and identically distributed.

The first assumption acknowledges the fact that, in a given sector (say an MF fund specialised in energy or a VC fund specialised in telecommunication), opportunities to invest or disinvest arise during common economic cycles. The second assumption takes into consideration the fact that two funds with identical investment strategies and created (for the sake of argument) at the same time but with different commitment periods and lifetimes, though subject to the same economic cycles as long as they co-exist, would not react to investment or disinvestment opportunities with the same magnitude whenever these arise. The fund with the shorter commitment period or the shorter lifetime must react with a greater intensity. The third and the fourth assumptions are quite standard for people used to dealing with multiple assets: their objective is to reduce the expression of the fund performance in terms of a limited number of principal factors (say l , with $l \leq m \leq n$), one being idiosyncratic, the others being market indices (like the NASDAQ or the S&P500 on the US market). Finally, the last assumptions is essentially a practical one. Thus, for a fund i of type k , the simplified model that can be built writes

$$\begin{aligned} dV_t^{(i)} &= [dD_t^{(i)}/dt + (\mu_t^{(k)} - \rho_t^{(i)})V_t^{(i)}]dt + \sigma_t^{(k)}V_t^{(i)} \sum_{j=1}^{l < n} \alpha^{(i,j)} dW_t^{\mu(j)}, \\ d\delta_t^{(i)} &= (c_1^{(i)} + c_2^{(i)}\delta_t^{(i)})dt + c_3^{(i)}\sqrt{\delta_t^{(i)}}(\sqrt{1 - \beta^{(k)^2}}dW_t^{\delta(i)} + \beta^{(k)}dW_t^{\delta(k)}), \\ d\rho_t^{(i)} &= (q_1^{(i)} + q_2^{(i)}\rho_t^{(i)})dt + q_3^{(i)}\sqrt{\rho_t^{(i)}}(\sqrt{1 - \gamma^{(k)^2}}dW_t^{\rho(i)} + \gamma^{(k)}dW_t^{\rho(k)}), \end{aligned} \quad (4.2)$$

where $\sum_{1 \leq j \leq l} \alpha_t^{(i,j)^2} = 1$. Besides, assuming that the initial scale that was used for inferring the diffusion coefficients was $T_*^{(k)} - T_{\bullet}^{(k)}$ for the drawdowns and $T^{(k)} - T_{\bullet}^{(k)}$ for the return payments, the parameters of the drawdown and repayment SDEs are defined^e as

$$\begin{aligned} c_1^{(i)} &= c_1^{(k)} \left(\frac{T_*^{(k)} - T_{\bullet}^{(k)}}{T_*^{(i)} - T_{\bullet}^{(i)}} \right)^2, & q_1^{(i)} &= q_1^{(k)} \left(\frac{T^{(k)} - T_{\bullet}^{(k)}}{T^{(i)} - T_{\bullet}^{(i)}} \right)^2, \\ c_2^{(i)} &= c_2^{(k)} \left(\frac{T_*^{(k)} - T_{\bullet}^{(k)}}{T_*^{(i)} - T_{\bullet}^{(i)}} \right), & q_2^{(i)} &= q_2^{(k)} \left(\frac{T^{(k)} - T_{\bullet}^{(k)}}{T^{(i)} - T_{\bullet}^{(i)}} \right), \\ c_3^{(i)} &= c_3^{(k)} \left(\frac{T_*^{(k)} - T_{\bullet}^{(k)}}{T_*^{(i)} - T_{\bullet}^{(i)}} \right), & q_3^{(i)} &= q_3^{(k)} \left(\frac{T^{(k)} - T_{\bullet}^{(k)}}{T^{(i)} - T_{\bullet}^{(i)}} \right). \end{aligned} \quad (4.3)$$

^eFor a given time interval T , the reference scales of the diffusion coefficients are $1/T^2$ for c_1 and q_1 , $1/T$ for c_2 and q_2 and $1/T$ for c_3 and q_3 .

The effect of rescaling the parameters of the diffusion process can be seen on the following charts. In the present case, they represent the randomly generated paths of the drawdown rate, cumulated drawdowns and drawdown flows. The number in brackets gives the ratio of the original time scale to the new time scale, i.e. $(T_*^{(k)} - T_\bullet^{(k)}) / (T_*^{(i)} - T_\bullet^{(i)})$.

Let us stress that still having different drawdown and repayment diffusion parameters according to the type of fund is not superfluous. Effectively, the analysis of Venture Economics data tend to show that, compared to BO or MF funds, drawdowns take place earlier and repayments later in VC funds hence shifting away the breakeven time. Also, it seems to be a bit difficult to ascertain the value of the correlation coefficients $\beta^{(k)}$ and $\gamma^{(k)}$, $1 \leq k \leq m$, and of the estimation error ε of each fund. In practice, they can actually be viewed as risk management parameters and utilised as a rational way to construct profit reserves. In particular, the coefficients $\beta^{(k)}$ and $\gamma^{(k)}$ somehow control the pool diversification: the smaller the correlation, the greater the diversification in the drawdown and repayment policies, which ensures more regular cash flows; conversely, the larger the correlation, the poorer the diversification, which can lead to extremely irregular cash flows.

For the estimation of the parameters of the fund performance, we assume that the investment volatility σ_t is constant throughout time but that the investment return μ_t is actually the addition of a time-variable risk-free rate r_t and of a risk premium ν_t that is itself constant throughout time. Since the risk-free rate can be independently observed (we can take either a one-month or a three-month LIBOR according to the observation frequency), only σ_t and ν_t need to be estimated.

The estimation of the drawdown parameters can be carried out separately since we have time series on both the committed capital and the fund NAV. Also, given that the drawdown transition distribution is known explicitly (it is the non-central chi-squared distribution), the parameter estimation is easier. As for the estimation of the other diffusion parameters, the immediate practical difficulty comes from the fact that PEF data are only observed quarterly or monthly at most. It is well known that standard maximum likelihood estimation (MLE) methods based on a Euler discretisation of the SDE are strongly biased when the time step is too large. To account for the very coarse observation frequency of PEF data, we use a MLE approach applied to the transition density function that is derived from the Milstein [15] approximation scheme instead of the standard Euler scheme. See [16, 17] and Appendix C for a quick insight. For more on the subject of diffusion parameter estimation in the context of discretely sampled observations, the interested reader can refer to [18]. Since even under the Milstein scheme, the parameter estimation would be biased, albeit less than under a Euler scheme, we actually combine the transition density implied by the Milstein scheme with the approach introduced by Pedersen [20]. The principle is to treat the values on a path connecting two discrete observations as missing and to introduce auxiliary points and discretise the process in-between. The likelihood function to maximise in the context of a Milstein discretisation is a non-central chi-squared density. It takes the actual observations

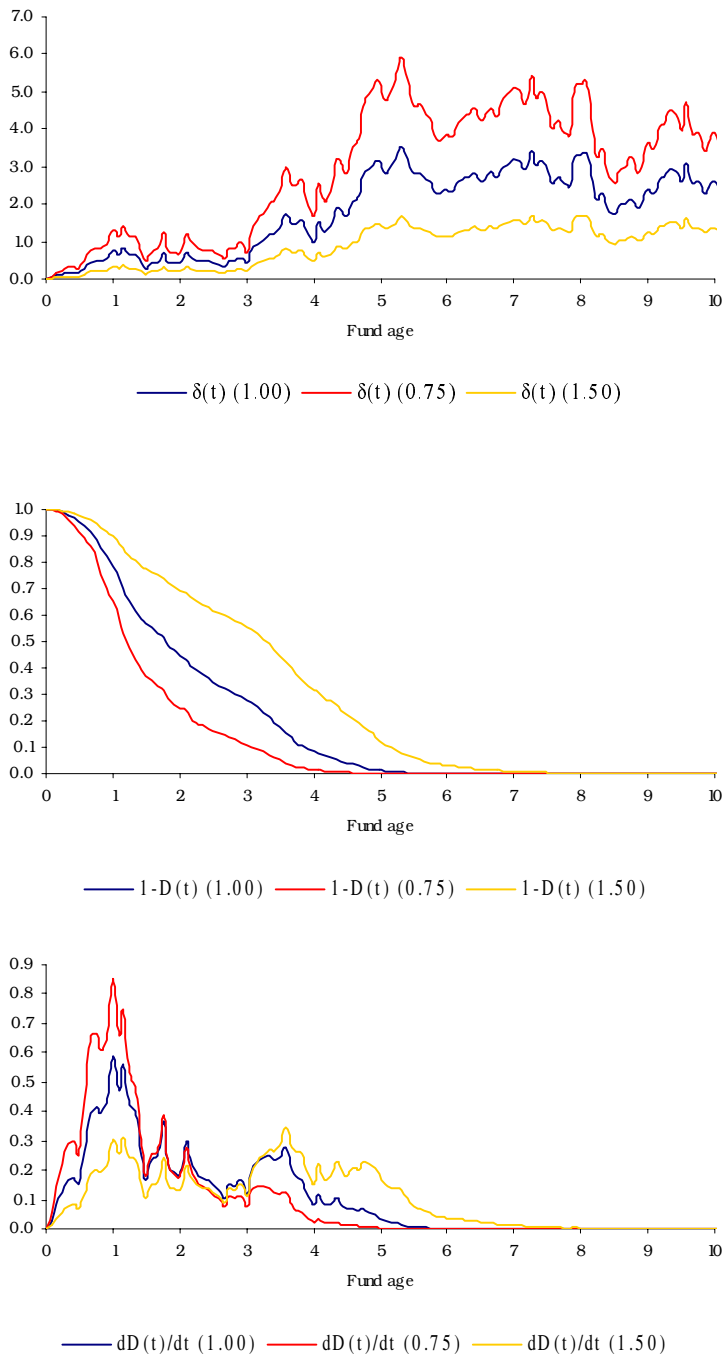


Fig. 5. Simulated drawdown process for different investment time scales. The original set of parameters used for the scale equal to (1.00) is: $c_1 = 0.35$, $c_2 = -0.10$, $c_3 = 0.60$. The other parameters are constructed following Eq. (4.3). The different paths are generated from the same set of normally distributed variables.

and the auxiliary points generated via Monte Carlo. In the context of the PEF model, the Milstein scheme approximation of the various equations that is used for the parameter estimation and the generation of the auxiliary points writes

$$\begin{aligned} V_{(i+1)\Delta} &= V_{i\Delta} + \delta_{i\Delta} D_{i\Delta} \Delta + (r_{i\Delta} + \nu - \rho_{i\Delta}) V_t^{(i)} \Delta + \sigma V_{i\Delta} \sqrt{\Delta} (U_{i+1}^\mu - U_i^\mu) \\ &\quad + \frac{1}{2} \sigma^2 V_{i\Delta} \Delta [(U_{i+1}^\mu - U_i^\mu)^2 - 1], \\ \delta_{(i+1)\Delta} &= c_1 \Delta + \delta_{i\Delta} (1 + c_2 \Delta) + c_3 \sqrt{\delta_{i\Delta} \Delta} (U_{i+1}^\delta - U_i^\delta) \\ &\quad + \frac{1}{4} c_3^2 \Delta [(U_{i+1}^\delta - U_i^\delta)^2 - 1], \\ \rho_{(i+1)\Delta} &= q_1 \Delta + \delta_{i\Delta} (1 + q_2 \Delta) \\ &\quad + q_3 \sqrt{\rho_{i\Delta} \Delta} (U_{i+1}^\rho - U_i^\rho) + \frac{1}{4} q_3^2 \Delta [(U_{i+1}^\rho - U_i^\rho)^2 - 1], \end{aligned} \quad (4.4)$$

where $U_{i+1}^\mu - U_i^\mu$, $U_{i+1}^\delta - U_i^\delta$ and $U_{i+1}^\rho - U_i^\rho$ are standard normal variables. Note that this discretisation scheme was also the one used to generate the various paths on Fig. 5. Some results of the estimation exercise are presented on Table 1 for VC and BO funds of different vintages.

Note that we sometimes found the estimated volatility coefficients c_3 and q_3 quite large compared to the drift coefficients c_1 and q_1 , respectively. In particular, the resulting values sometimes allow the relations $c_1 > c_3^2/2$ and $q_1 > q_3^2/2$ to break down, which in theory makes possible the occurrence of negative drawdown or repayment rates. It comes from the high drawdown and repayment rates that can be observed towards the end of the investment period T_* and the fund lifetime T , respectively. The reason is that the underlying committed capital and the fund value have an exponential decay and the behaviour of the drawdown and repayment rates tend to become more erratic as these values get close to zero. The next two charts (Fig. 6) compare some actual data to the expectations that can be constructed from

Table 1. Parameter estimation for VC and BO private equity funds.

Type	VC	VC	BO
Dates	$T_\bullet = 1986$	$T_\bullet = 1988$	$T_\bullet = 1987$
	$T_* = 1991$	$T_* = 1998$	$T_* = 2002$
	$T = 2001$	$T = 2008$	$T = 2007$
c_1	2.180043	0.623261	2.821435
c_2	-3.045790	-0.742688	-8.739792
c_3	1.864828	0.999960	1.462722
q_1	0.143379	0.262438	3.508246
q_2	-0.596600	-0.650971	-17.467964
q_3	0.761356	1.376160	1.929698
ν	0.039821	0.019959	0.042838
σ	0.322001	0.167343	0.293718

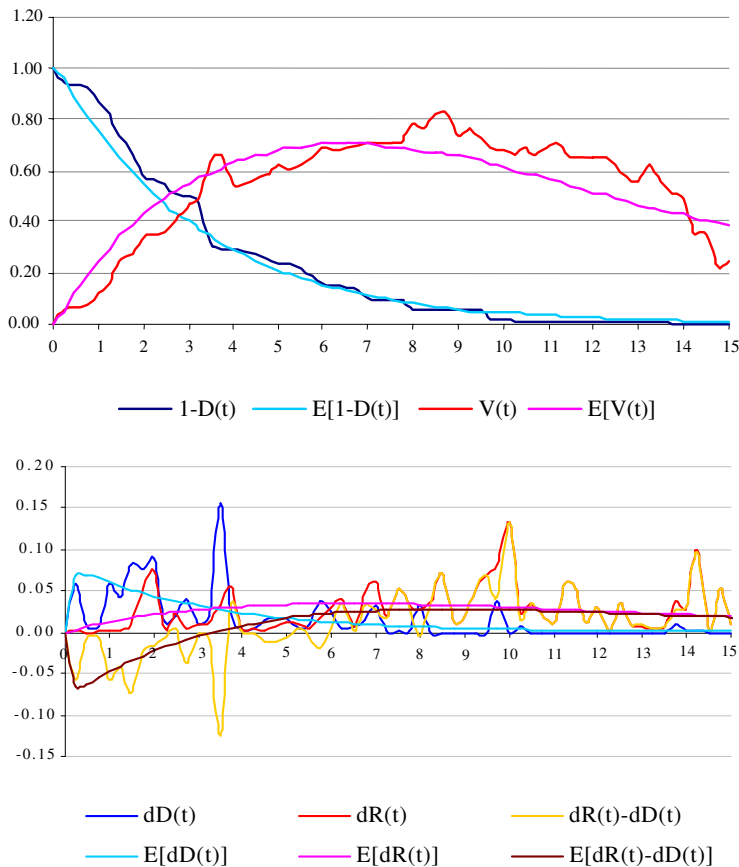


Fig. 6. Actual vs. expected commitment, drawdowns, repayments and fund value. The parameters used to calculate the expectations are those reported on Table 1 for the BO fund.

the estimated diffusion parameters. The data are those of the BO fund of Table 1. It gives a rough idea of the goodness of fit.

We finally illustrate the model by calculating the expected future cash flows produced by a fund of funds. The detail of the portfolio is given in Appendix D and mainly consists of BO funds invested in equity. The individual funds have all started, which means that only a fraction of the committed capital is still undrawn and the NAV is already greater than zero. The parameters characterising the performance of the funds are taken from Tremont hedge fund database. Two series of charts are produced. The first series (Fig. 7) reports the total undrawn capital and the total NAV of the fund of funds and the drawdown and repayment flows.

On the second series (Fig. 8) of graphs, we have measured and represented the sensitivity of the cash flows (drawdowns and return payments) to the parameters of the model. We see that the expected cash flows (drawdowns and repayments) are accelerated whenever the drift parameters c_1 , c_2 , q_1 and q_2 increase. Conversely,

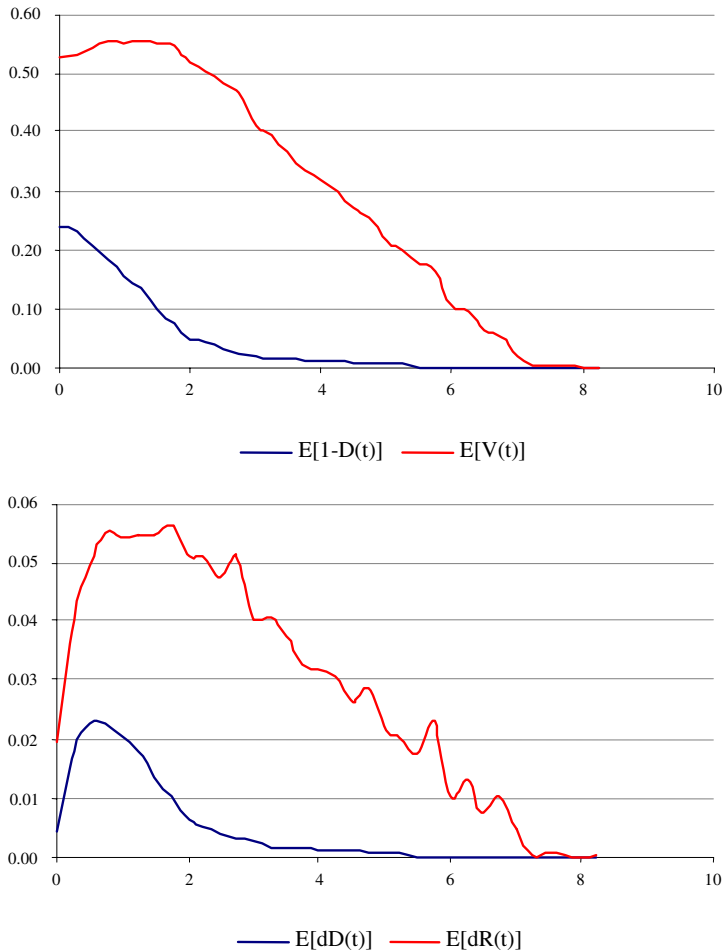


Fig. 7. Expected commitment, value, drawdown flows and repayment flows of a fund of funds.

these cash flows slow down and are shifted away in time when the volatility coefficients c_3 and q_3 increase.

5. Conclusion

Finding an appropriate dynamics for the drawdown and the repayment behaviours is the key question when building a PEF valuation model. The use of exogenous squared Bessel processes for drawdowns and repayments greatly facilitates the task. In such a framework, valuing a PEF boils down to finding the solution of an SDE. For more complex PEF-linked transactions, such a framework can be adequately utilised to build a Monte Carlo engine that will provide prices and risk figures for the trades in question. The drawback is that the model relies on diffusion parameters

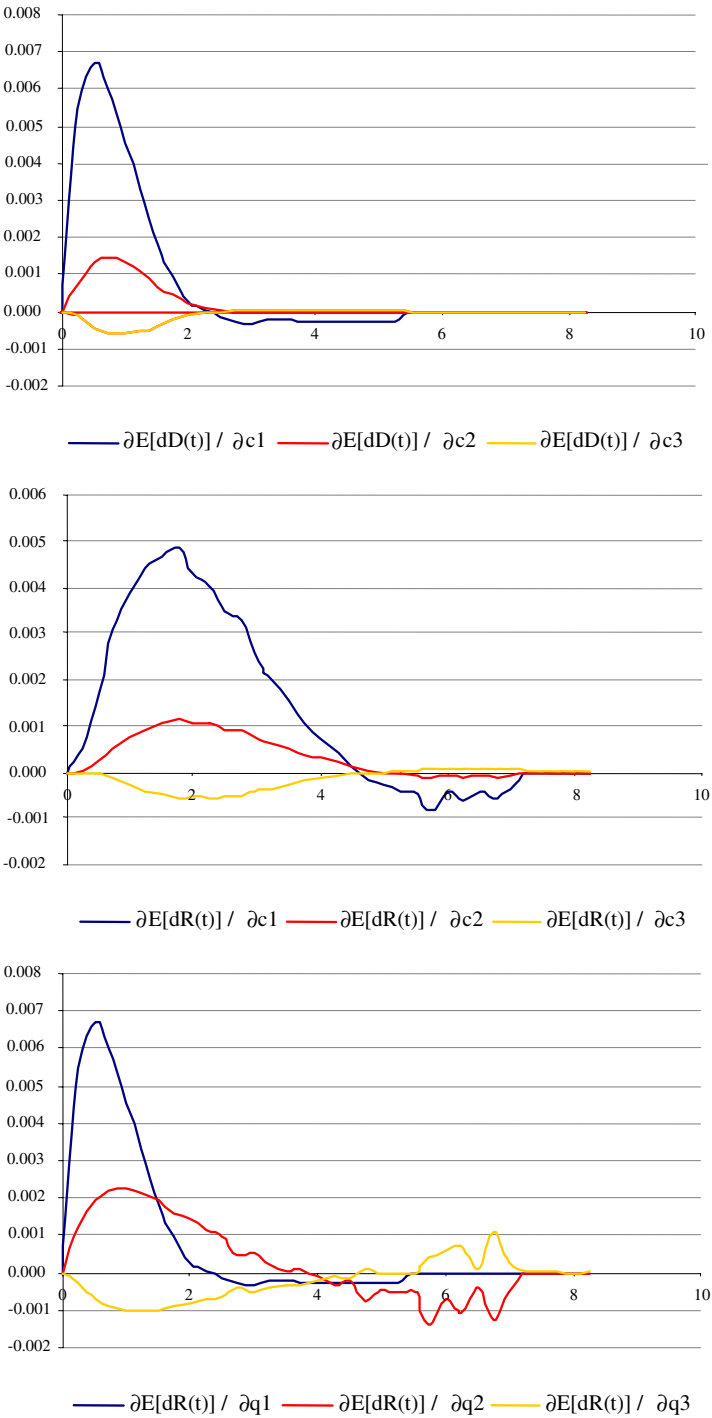


Fig. 8. Sensitivity of the fund of funds drawdowns and repayment flows to the model parameters.

that are neither tradable nor observable on the market. The parameters have indeed to be estimated from historical data. Thus any PEF-linked transaction would be difficult to hedge. Yet, the model can be used as a rational way to build confidence bounds around the trade fair price and to put in place profit reserves. In practice, the model seems to fit quite well to historical data. A natural extension would be to include some jumps that could be helpful to tackle some of the extreme cash flow movements that are sometime observed. The model would lose in parsimony but not in tractability.

Appendix A

A.1. Fund Sectors and Strategies

The “sectors” reported on Table 2 were established by Thomson Financial/Venture Economics. Venture Economics private equity database has tracked the quarterly performance of about 1,500 US venture capital and buy-out funds formed since 1969.

The various “strategies” reported on Table 3 are those followed by Crédit Suisse First Boston/Tremont, for which it has been possible to have monthly data on returns, volatilities and Sharpe ratios (risk-adjusted excess return with respect to the three-month treasury bill rate) since 1993.

A.2. Examples of Deterministic Cash Flow Funds

Example 1. All rates δ_t , μ_t and ρ_t are constant equal to δ , μ and ρ respectively and the volatility σ_t is zero. The solution to the equation is

$$V_t = \begin{cases} \frac{\delta}{\rho - \mu - \delta} [e^{-\delta t} - e^{(\mu - \rho)t}], & \text{if } \rho - \mu - \delta \neq 0, \\ \delta t e^{(\mu - \rho)t}, & \text{if } \rho - \mu - \delta = 0. \end{cases}$$

Example 2. The yield μ_t is constant equal to μ whereas the drawdowns δ_t and the repayment ρ_t are a linear function of the time, i.e. $\delta_t = \delta t$ and $\rho_t = \rho t$ with $\delta \geq \rho > 0$. The volatility σ_t is still zero. The solution to the equation is

$$V_t = \begin{cases} \frac{\delta}{\delta - \rho} \left[e^{\mu t - \rho t^2 / 2} - e^{-\delta t^2 / 2} - \frac{\mu \sqrt{2\pi}}{\sqrt{\delta - \rho}} e^{\frac{\mu^2}{2(\delta - \rho)} + \mu t - \rho t^2 / 2} \right. \\ \quad \times \left[N \left(\sqrt{\delta - \rho} t + \frac{\mu}{\sqrt{\delta - \rho}} \right) - N \left(\frac{\mu}{\sqrt{\delta - \rho}} \right) \right] \Bigg], & \text{if } \delta > \rho, \\ \frac{\delta}{\mu^2} e^{\mu t - \rho t^2 / 2} G_{2,\mu}(t), & \text{if } \delta = \rho. \end{cases}$$

In the equation, $N(x)$ is the standard normal cumulative function and $G_{a,b}(x)$, with

Table 2. Private equity fund sectors.

VC	Venture capital
VCSS	• Seed/early stage
VCMS	• Medium/balanced stage
VCLS	• Later/expansion stage
BO	Buy-out
MF	Mezzanine finance
SP	Secondary purchase
RE	Real estate
DI	Direct investment

Table 3. Hedge fund strategies and asset classes.

CA	Convertible arbitrage	Equity + debt
EQ	Equity-dedicated	Equity
EQLS	• Long-short	
EQMN	• Market neutral	
EQSB	• Short biased	
EM	Emerging markets	Equity + debt
EV	Event-driven	Equity + debt
EVDC	• Distressed companies	
EVMS	• Multi-strategy	
EVRA	• Risk arbitrage	
FA	Fixed income arbitrage	Debt
GM	Global macro	Equity + debt
MF	Managed futures	Equity + debt + commodity

parameters $a > 0$ and $b > 0$, is the gamma cumulative function given by

$$G_{a,b}(x) = \frac{b}{\Gamma(a)} \int_0^x (bt)^{a-1} e^{-bt} dt,$$
$$\text{with } \Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t} dt.$$

A.3. Venture Economics Data

Charts of the same nature as the deterministic ones illustrated by Figs. 1 and 2 are presented hereafter, namely the drawdowns and repayments flows as well as the cumulated drawdowns and repayments actually experienced by mixtures of BO and MF funds of different vintages. This shows, in spite of the randomness factor that is not accounted for in the deterministic funds of Appendix A.2 (Figs. 1 and 2), that our models are suitable for the analysis (especially the one that makes the drawdowns and repayments linear functions of time). All flows are normalised to

the total drawn committed amount.

- Vintage 1985:

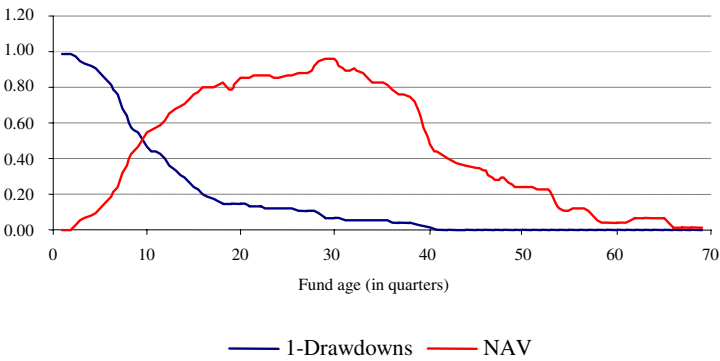


Fig. 9. Undrawn commitment and fund value.

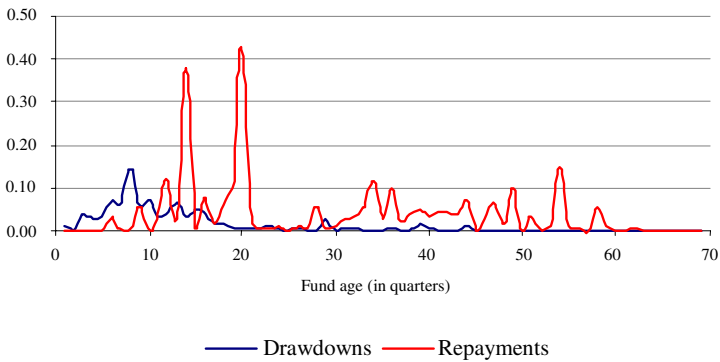


Fig. 10. Drawdown and repayment flows.

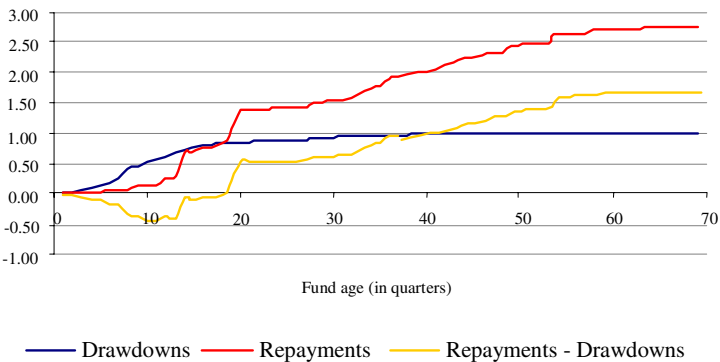


Fig. 11. Cumulated drawdowns and repayments.

• Vintage 1987:

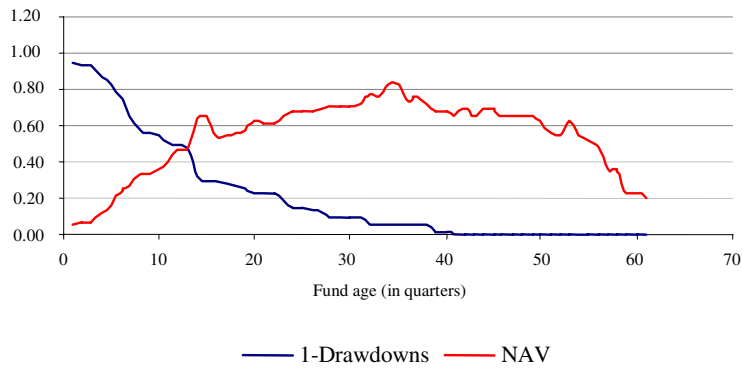


Fig. 12. Undrawn commitment and fund value.

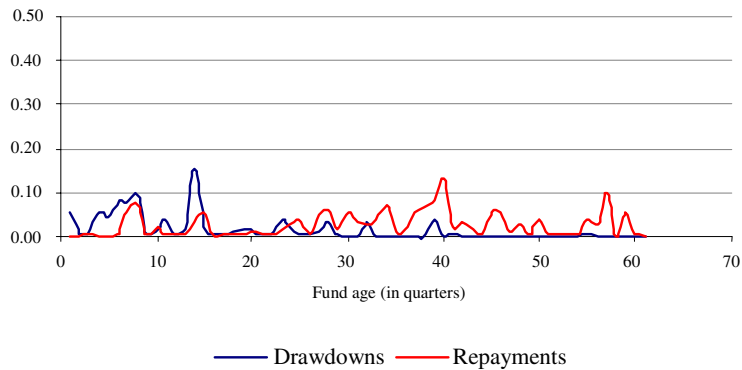


Fig. 13. Drawdown and repayment flows.

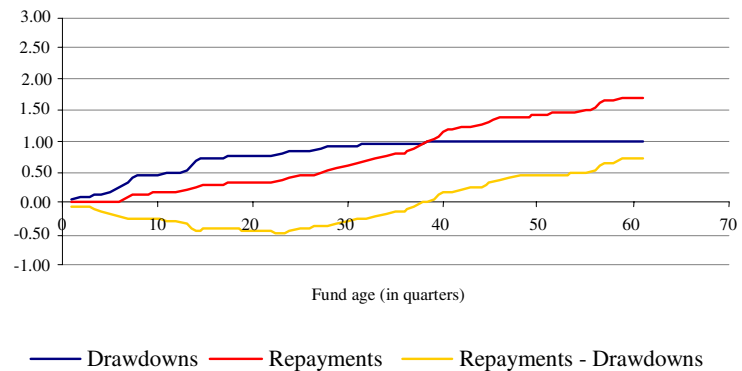


Fig. 14. Cumulated drawdowns and repayments.

• Vintage 1989:

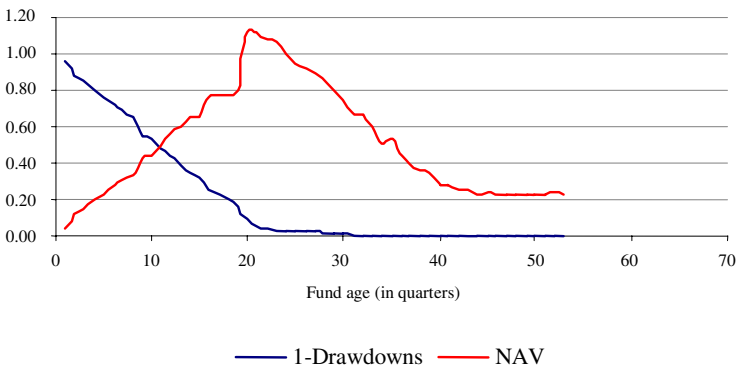


Fig. 15. Undrawn commitment and fund value.

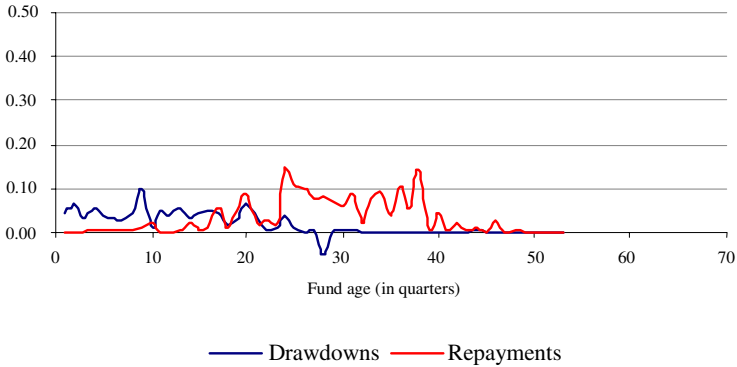


Fig. 16. Drawdown and repayment flows.

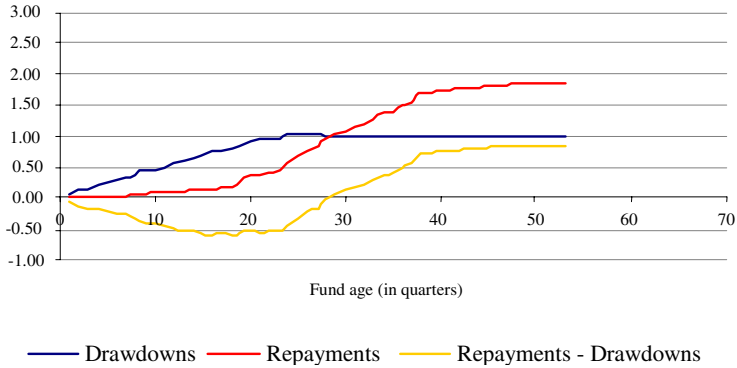


Fig. 17. Cumulated drawdowns and repayments.

Appendix B

B.1. Laplace Transform of a Squared Bessel Process

The following one-dimensional SDE

$$dX_t = (a + bX_t)dt + c\sqrt{X_t}dW_t. \quad (\text{B.1})$$

with a standard Brownian motion W_t and real constants a , b and c such that $c > 0$ defines X_t as a so-called squared Bessel process. It is well known (see e.g. [5] or [7]) that the process X_t starting from $X_0 = x$ has a non-central chi-squared distribution. In particular, the Laplace transform of the process X_t starting from x for a positive real number λ is

$$\mathbb{E}[e^{-\lambda X_t} | X_0 = x] = \left[\frac{c^2 \lambda}{2b} (e^{bt} - 1) + 1 \right]^{-2a/c^2} \exp \left(-\frac{\lambda e^{bt} x}{\frac{c^2 \lambda}{2b} (e^{bt} - 1) + 1} \right). \quad (\text{B.2})$$

It is understood that for $b = 0$, $(e^{bt} - 1)/b = t$.

B.2. A Particular Feynman–Kac Functional of a Squared Bessel Process

The following equation allows the characterisation of the joint distribution of $(X_t, \int_0^t X_s ds)$, which is an extremely useful result when using squared Bessel processes. It is more general than the result obtained in Appendix B.1. It can be found for example in [19]. For any positive real numbers λ and μ , we have

$$\mathbb{E} \left[e^{-\lambda X_t - \mu \int_0^t X_s ds} | X_0 = x \right] = e^{-F_{\lambda, \mu}(t) - G_{\lambda, \mu}(t)x}, \quad (\text{B.3})$$

where the two functions $F_{\lambda, \mu}(t)$ and $G_{\lambda, \mu}(t)$ are given by

$$\begin{aligned} F_{\lambda, \mu}(t) &= -\frac{2a}{c^2} \ln \left(\frac{2\xi e^{-bt/2}}{2\xi \cosh \xi t + (c^2 \lambda - b) \sinh \xi t} \right), \\ G_{\lambda, \mu}(t) &= \frac{2\lambda \xi \cosh \xi t + (b\lambda + 2\mu) \sinh \xi t}{2\xi \cosh \xi t + (c^2 \lambda - b) \sinh \xi t}, \end{aligned} \quad (\text{B.4})$$

with $2\xi = \sqrt{b^2 + 2\mu c^2}$. The fact that the expectation has the form of an exponential of an affine function of x is a result of the additivity of the process X_t with respect to the initial condition x . This property is explicit in the shape of the Laplace transform of X_t presented in Appendix B.1. As a consequence of the Feynman–Kac formula and presuming a solution of an exponential type, it is possible to check that the functions $F_{\lambda, \mu}(t)$ and $G_{\lambda, \mu}(t)$ are actually solving

$$\begin{cases} \dot{G}_{\lambda, \mu}(t) = -\frac{c^2}{2} G_{\lambda, \mu}^2(t) + bG_{\lambda, \mu}(t) + \mu \\ \dot{F}_{\lambda, \mu}(t) = aG_{\lambda, \mu}(t) \end{cases} \quad (\text{B.5})$$

with $F_{\lambda,\mu}(0) = 0$, $G_{\lambda,\mu}(0) = \lambda$, $\dot{F}_{\lambda,\mu}(t) = \partial F_{\lambda,\mu}(t)/\partial t$ and $\dot{G}_{\lambda,\mu}(t) = \partial G_{\lambda,\mu}(t)/\partial t$. Note also the very straightforward properties:

$$\begin{aligned} F_{\lambda,\mu}(t; a, b, c) &= F_{\lambda/\mu,1}(t; \mu a, b, \sqrt{\mu}c), \\ G_{\lambda,\mu}(t; a, b, c) &= \mu G_{\lambda/\mu,1}(t; \mu a, b, \sqrt{\mu}c), \\ F_{\lambda,0}(t; a, b, c) &= F_{1,0}(t; \lambda a, b, \sqrt{\lambda}c), \\ G_{\lambda,0}(t; a, b, c) &= \lambda G_{1,0}(t; \lambda a, b, \sqrt{\lambda}c). \end{aligned} \quad (\text{B.6})$$

Formula (B.3) can also be utilised for the calculation of the moments of a functional of X_t of particular interest. We can indeed construct the variable

$$Y_t = X_t e^{-\int_0^t X_s ds} = -\frac{d}{d\lambda} e^{-\lambda X_t - \int_0^t X_s ds} \Big|_{\lambda=0}. \quad (\text{B.7})$$

Since, by dominated convergence, we have

$$\mathbb{E}_x \left[\frac{d}{d\lambda} e^{-\lambda X_t - \int_0^t X_s ds} \right] = \frac{d}{d\lambda} \mathbb{E}_x \left[e^{-\lambda X_t - \int_0^t X_s ds} \right], \quad (\text{B.8})$$

it is fairly easy from Eq. (B.3) to compute $\mathbb{E}_x[Y_t]$. Following the same route, we have

$$Y_t^m = X_t^m e^{-m \int_0^t X_s ds} = (-1)^m \frac{d^m}{d\lambda^m} e^{-\lambda X_t - m \int_0^t X_s ds} \Big|_{\lambda=0}. \quad (\text{B.9})$$

Thus it is possible to obtain all the moments of Y_t by simple derivation.

Appendix C

The Milstein scheme is essentially a second-order Taylor approximation of the SDE and hence provides a better approximation than the Euler scheme that is a first-order Taylor approximation. Assuming a set of observations at times $0, \dots, i\Delta, \dots, n\Delta$, the Milstein scheme of a general SDE $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ takes the form

$$\begin{aligned} X_{(i+1)\Delta} &= X_{i\Delta} + \mu(X_{i\Delta})\Delta + \sigma(X_{i\Delta})\sqrt{\Delta}(U_{i+1} - U_i) \\ &\quad + \frac{1}{2}\sigma(X_{i\Delta})\sigma'(X_{i\Delta})\Delta[(U_{i+1} - U_i)^2 - 1], \end{aligned} \quad (\text{C.1})$$

where $(U_{i+1} - U_i)$ is a normal distribution with mean 0 and variance 1. The Milstein scheme is the Euler scheme plus an additional term but it reduces to the Euler scheme whenever $\sigma'(X_{i\Delta}) = d\sigma(X_{i\Delta})/dX_{i\Delta} = 0$.

In the general case, when $\sigma'(X_{i\Delta}) \neq 0$, whilst the Euler scheme implies a normal transition density with mean $X_{i\Delta} + \mu(X_{i\Delta})\Delta$ and variance $\sigma^2(X_{i\Delta})\Delta$, the Milstein scheme implies a non-central chi-squared transition density. See [16]. The transition density function $p(X_{(i+1)\Delta}|X_{i\Delta})$ is given by

$$p(X_{(i+1)\Delta}|X_{i\Delta}) = \frac{\exp(-C/2)}{|A|\sqrt{2\pi}} Y_{(i+1)\Delta}^{-1/2} \exp(-Y_{(i+1)\Delta}/2) \cosh(\sqrt{CY_{(i+1)\Delta}}), \quad (\text{C.2})$$

Table 4. Pool of PEFs.

#	Strategy		$T_* - T_\bullet$	$T - T_\bullet$	$(T_* - t)^+$	$T - t$	Fees %	δ_t	ρ_t	w %	$1 - D_t$	V_t
1	BO	EQ	6.00	9.76	0.00	0.59	1.50	0.000	0.224	1.5522	0.033074	0.603154
2	SP	EQ	5.47	10.01	0.00	2.08	1.50	0.000	0.316	1.8586	0.002198	0.402863
3	BO	EQ	5.00	10.01	0.00	1.62	2.00	0.008	0.042	4.6681	0.121284	0.331520
4	BO	EQ	5.01	10.01	0.00	3.25	2.00	0.007	0.000	0.9505	0.069791	0.851500
5	BO	EQ	4.02	8.52	0.00	2.59	1.50	0.012	0.000	4.2332	0.022568	0.819247
6	BO	EQ	5.01	10.01	0.00	3.23	1.50	0.000	0.000	2.8172	0.037957	0.537838
7	BO	EQ	6.01	10.01	0.00	3.41	2.00	0.003	0.000	1.3055	0.144749	0.519114
8	BO	EQ	4.52	9.52	0.00	3.86	2.00	0.034	0.809	1.0273	0.360520	0.362252
9	BO	EQ	5.84	10.28	1.13	5.56	2.00	0.149	0.270	2.8324	0.292534	0.555544
10	BO	EQ	4.29	9.30	1.33	6.34	1.50	0.009	0.252	2.2577	0.576427	0.289115
11	BO	EQ	6.00	10.44	1.15	5.59	2.00	0.183	0.355	2.8197	0.156001	0.740470
12	BO	EQ	5.48	10.12	0.59	5.23	1.75	0.062	0.740	0.9877	0.048785	0.764485
13	BO	EQ	4.32	9.33	0.00	4.61	1.50	0.139	0.000	7.5069	0.306298	0.591589
14	BO	EQ	5.00	10.01	0.33	5.33	2.00	0.021	0.000	0.4233	0.247669	0.469844
15	BO	EQ	6.00	10.01	0.07	4.08	1.50	0.004	1.252	2.6105	0.011097	0.743911
16	BO	EQ	5.65	9.65	0.76	4.77	1.34	0.000	0.000	2.0724	0.170380	0.441314
17	BO	EQ	5.57	10.64	0.51	5.59	1.50	0.000	0.986	1.1564	0.651230	0.272289
18	BO	EQ	1.28	7.29	0.00	3.83	2.50	0.074	0.000	0.8607	0.135335	0.714804
19	BO	EQ	4.03	9.03	0.24	5.24	2.00	0.109	0.106	0.4876	0.081690	0.464323
20	BO	EQ	5.52	10.53	1.58	6.59	2.00	0.068	0.541	2.5399	0.053866	0.807388
21	BO	EQ	5.52	10.53	1.58	6.59	2.00	0.008	0.089	0.8940	0.182214	0.927077
22	BO	EQ	6.01	10.01	1.91	5.91	2.00	0.066	0.002	0.4290	0.137382	0.656230
23	BO	EQ	4.63	8.11	1.11	4.59	2.50	0.500	0.000	0.1613	0.213739	0.478995

Table 4. (*Continued*)

#	Strategy		$T_* - T_\bullet$	$T - T_\bullet$	$(T_* - t)^+$	$T - t$	Fees %	δ_t	ρ_t	w %	$1 - D_t$	V_t
24	BO	EQ	5.00	10.01	1.38	6.39	1.75	0.055	0.000	2.5656	0.382877	0.379825
25	VC	EQ	5.92	9.93	1.93	5.93	1.75	0.060	0.572	0.8466	0.126471	0.534942
26	BO	EQ	6.01	10.01	2.08	6.08	2.00	0.072	0.000	3.0112	0.291458	0.843368
27	BO	EQ	5.00	10.01	1.23	6.24	2.00	0.041	0.003	0.6401	0.288848	0.519686
28	BO	EQ	5.13	10.13	1.00	6.01	2.00	0.027	0.000	4.2332	0.444955	0.375890
29	SP	EV	5.00	10.01	0.72	5.72	2.00	0.366	0.124	0.4536	0.245637	1.007332
30	BO	EQ	5.91	9.93	2.09	6.11	2.00	0.116	0.000	0.4048	0.293829	0.646068
31	BO	EQ	6.00	10.01	1.45	5.45	2.00	0.031	0.012	2.1451	0.253011	0.427977
32	BO	EQ	5.50	10.50	1.97	6.97	1.50	0.077	0.037	2.4835	0.239477	0.608313
33	BO	EQ	5.52	9.53	1.60	5.61	1.50	0.020	0.000	8.4668	0.030000	0.699866
34	VC	EQ	6.01	10.01	2.17	6.17	2.00	0.052	0.000	0.7610	0.080630	0.462423
35	VC	EQ	5.00	10.01	1.54	6.55	2.00	0.041	0.005	1.0160	0.117136	0.459971
36	BO	EQ	4.00	8.01	0.99	4.99	2.00	0.078	0.000	0.3063	0.429304	0.599307
37	MF	EQ	5.60	9.60	2.77	6.77	1.50	0.164	0.441	1.2700	0.242468	0.596855
38	BO	EQ	5.25	10.25	2.44	7.44	2.00	0.030	0.000	0.5700	0.232857	0.854363
39	BO	EQ	6.01	9.95	3.42	7.37	2.00	0.432	0.007	0.4445	0.373589	0.534701
40	BO	EQ	5.00	10.01	1.62	6.63	2.00	0.198	0.120	8.2470	0.444349	0.410960
41	BO	EQ	5.01	8.01	2.04	5.04	2.00	0.027	0.000	1.5834	0.350577	0.551402
42	VC	EQ	4.07	6.07	0.67	2.67	1.50	0.212	0.000	0.5318	0.230264	0.361462
43	BO	EQ	6.01	10.01	2.30	6.30	2.00	0.000	0.018	0.6542	0.580500	0.426795
44	BO	EQ	4.89	8.89	2.97	6.97	1.75	0.016	0.000	6.3440	0.447712	0.405946
45	SP	EQ	4.00	9.01	1.81	6.82	1.75	0.145	0.012	0.2424	0.445838	0.523066
46	BO	EQ	7.00	10.01	5.08	8.08	2.00	0.026	0.198	5.9265	0.925019	0.072285
47	BO	EQ	6.01	10.01	3.74	7.74	2.00	0.000	0.000	0.4007	0.421026	0.576853

where

$$\begin{aligned} Y_{(i+1)\Delta} &= \frac{X_{(i+1)\Delta} - B}{A}, \\ A &= \frac{1}{2}\sigma(X_{i\Delta})\sigma'(X_{i\Delta})\Delta, \\ B &= -\frac{1}{2}\frac{\sigma(X_{i\Delta})}{\sigma'(X_{i\Delta})} + X_{i\Delta} + \mu(X_{i\Delta})\Delta - A, \\ C &= \frac{1}{\sigma'^2(X_{i\Delta})\Delta}. \end{aligned}$$

Appendix D

The composition of the pool of PEFs at the time t of the analysis is given on the next table. Note that the drawdown rate δ_t and distribution rate ρ_t are those that were observed for each individual fund just before closing the deal. Actually they are individually estimated as the sum of drawdowns and repayments experienced over the previous quarter divided by the total commitment and the last available NAV respectively. Note also that the residual committed capital and the NAV of each individual fund are scaled to the initial committed capital.

The diffusion parameters are reported on Table 5 and Table 6. Depending on the fund strategy, the parameters have to be picked up from the relevant row and column. For the re-scaling, we have taken an original scale equal to 10 for the drawdown parameters and equal to 20 for the distribution parameters.

The three-month risk-free rate used for building the curves is assumed to be flat and equal to 5%. The net performance of a fund is equal to the risk-free rate plus the risk premium ν minus the management fees that are reported on Table 4.

Table 5. Drawdown and repayment parameters.

Type	VC	BO	MF	SP
c_1	2.180043	2.821435	2.821435	2.821435
c_2	-3.045790	-8.739792	-8.739792	-8.739792
c_3	1.864828	1.462722	1.462722	1.462722
q_1	0.143379	3.508246	3.508246	3.508246
q_2	-0.596600	-17.467964	-17.467964	-17.467964
q_3	0.761356	1.929698	1.929698	1.929698

Table 6. Fund risk premium and volatility.

Type	EQ	EV
ν	0.0629	0.0564
σ	0.1137	0.0621

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